A universal space for normal bundles of n-manifolds.

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§1. Introduction

In [3] the authors gave a simple criterion for deciding whether a polynomial in Stiefel–Whitney classes is zero on the normal bundles of all smooth n-manifolds. The ideal of relations among Stiefel–Whitney classes for all n-manifolds, $I_n \subset H^*(BO)$ was defined by

$$I_n = \{w \in H^*(BO) \mid w(\nu_{M^n}) = 0 \quad \text{for all} \quad M^n\}$$

where M^n denotes a smooth n-manifold and ν_M is its stable normal bundle. Let $\Phi : H^*(BO) \simeq H^*(MO)$ be the Thom isomorphism and for $w \in H^*(BO)$, define wSq^i to be $\Phi^{-1}(\chi(Sq^i)\Phi(w))$. It was shown that I_n consists of all Z_2-linear combinations of elements of the form wSq^i where $2i > n - |w|$ ($|w|$ = dimension of w).

In this paper we give a stronger version of this result, namely:

THEOREM 1. *There is a space BO/I_n and a map $\pi : BO/I_n \to BO$ such that*

(a) *If M is a smooth, compact n-manifold and $h : M \to BO$ classifies ν_M, then there is a map $\bar{h} : M \to BO/I_n$ such that $\pi\bar{h} \simeq h$.*

(b) *The following sequence is exact.*

$$0 \longrightarrow I_n \subset H^*(BO) \xrightarrow{\pi^*} H^*(BO/I_n) \longrightarrow 0.$$

Theorem 1 shows that BO/I_n is a universal space for normal bundles of n-manifolds in that stably, every such bundle is induced from the bundle over BO/I_n and BO/I_n is the space with the smallest cohomology having this property.

Our original result on I_n suggested the possibility of defining higher order characteristic classes, that is, one could form a space B over BO by killing the

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elements of I_n. Then an element of $H^*(B)$ might give a "new" characteristic class for n-manifolds. For example, with $n=4$ or 5, the relation

$$(Sq^2+w_1 \cup Sq^1+w_2U)(v_3)=v_3Sq^2=(1Sq^3)Sq^2=0$$

where v_3 is the *Wu* class, gives a class in $H^4(B)$ which is not a polynomial in Stiefel–Whitney classes. Theorem 1 shows that on an n-manifold this "new" class will be a polynomial in Stiefel-Whitney classes modulo indeterminacy.

The spaces BO/I_n are also related to the conjecture that any smooth n-manifold immerses in $R^{2n-\alpha(n)}$ where $\alpha(n)$ is the number of ones in the dyadic expansion of n. Since this conjecture is equivalent to the normal bundle map $h: M^n \to BO$ lifting to $BO_{n-\alpha(n)}$ ([9]), the following is a stronger form of the conjecture:

CONJECTURE. $\pi: BO/I_n \to BO$ lifts to $BO_{n-\alpha(n)}$.

Using our proof of Theorem 1, our results in [4] can be restated in the following way which gives some plausibility to the above conjecture.

THEOREM 2. *If ζ is the stable universal bundle over BO,* MO *is its Thom spectrum,* MO/I_n *is the Thom spectrum of* $\pi^*\zeta$ *and* $\mathrm{MO}(n-\alpha(n))$ *is the Thom spectrum of the universal bundle over* $BO_{n-\alpha(n)}$, *then* MO/I_n *lifts to* $\mathrm{M}O(n-a(n))$.

This paper is organized as follows: In §2 we give a detailed outline of the proof of Theorem 1 setting forth most of the notation and describing the various technical problems arising in the construction of BO/I_n. Then in Sections 3, 4, 5, and 6 we prove the various lemmas stated in §2. Throughout the remainder of this paper n is a fixed positive integer.

§2. Outline of the Proof of Theorem 1

All cohomology will be with Z_2 coefficients, A will be the mod two Steenrod algebra and $\chi: A \to A$ will be the canonical antiautomorphism. The semi-tensor product of A and $H^*(BO)$ ([6]) will be denoted by $A(BO)$, that is, $A(BO)= A \otimes H^*(BO)$ with the algebra structure defined by

$$(a \otimes u)(b \otimes v)=\sum ab_i' \otimes (\chi(b_i'')u)v$$

where $b \to \sum b_i' \otimes b_i''$ under the diagonal of A. We denote $a \otimes u$ by $a \circ u$.

By a spectrum Y, we will mean a collection of spaces Y_q and maps $g_q : SY_q \to Y_{q+1}$. If X and Y are spectra, a map $f : X \to Y$ of degree p will be a collection of homotopy classes $f_q \in [X_q, Y_{q+p}]$ compatible with the maps g_q. If ξ is a real k-plane bundle, $T(\xi)$ will denote its Thom spectrum, i.e., $T(\xi)_q = S^{q-k}$ (Thom space of ξ). Thus the Thom class is in $H^0(T(\xi))$. If ξ is a vector bundle over B, $\Phi : H^*(B) \approx H^*(T(\xi))$ will be the Thom isomorphism. We make $H^*(T(\xi))$ into an $A(BO)$ module as follows: Let $h : B \to BO$ classify ξ. If $u \in H^*(T((\xi))$, $w \in H^*(BO)$ and $a \in A$, $(a \circ w)u = a(h^*(w)u)$. One easily checks that $\Phi(I_n) \subset H^*(\mathrm{MO})$ is an $A(BO)$ submodule.

We begin by constructing an A-free, acyclic resolution of $\Phi(I_n)$. In [3] the following was proved:

THEOREM 2.1. *If $\{u_i\}$ is an A basis for $H^*(\mathrm{MO})$, then $\Phi(I_n)$ is the A module generated by*

$$\{\chi(Sq^j)u_i \mid 2j > n - |u_i|\}.$$

For a partition $\omega = \{j_1, j_2, \ldots, j_l\}$ let $s_\omega \in H^*(BO)$ be the usual class ([17]) associated with the symmetric function $\sum t_1^{j_1} t_2^{j_2} \cdots t_l^{j_l}$. For each partition ω let ω_r be the partition consisting of odd integers j, one for each $j2^r \in \omega$. Let

$$u_\omega = \prod_r s_{\omega_r}^{2^r}$$

Since

$$u_\omega = s_\omega + \sum s_{\omega'}$$

where ω' has fewer entries than ω and $\{s_\omega\}$ is a basis for $H^*(BO)$, $\{u_\omega\}$ is also a basis for $H^*(BO)$. Also $\{\Phi(u_\omega) \mid 2^i - 1 \notin \omega\}$ is an A basis for $H^*(\mathrm{MO})$ since $\{\Phi(s_\omega) \mid 2^i - 1 \notin \omega\}$ is.

In [2] an A-free acyclic resolution of $A/A\{\chi(Sq^i) \mid i > h\}$ was constructed. Combining these resolutions with 2.1 and the $\Phi(u_\omega)$ basis, we obtain the following resolution of $\Phi(I_n)$.

Let Λ be the graded free associative algebra over Z_2 with unit generated by λ_i, $i = 0, \pm 1, \pm 2, \ldots$, $|\lambda_i| = i$, modulo the relations: If $2i < j$

$$\lambda_i \lambda_j = \sum \binom{s-1}{2s-(j-2i)} \lambda_{i+s} \lambda_{j-s}.$$

If $I = (i_1, i_2, \ldots, i_l)$, let $\lambda_I = \lambda_{i_1} \lambda_{i_2} \cdots \lambda_{i_l}$, $l(I) = l$, $t(I) = i_l$, and $\lambda_{()} = 1$. We define I

to be admissible if $2i_j \geqq i_{j+1}$. As we will see in §3, $\{\lambda_I \mid I \text{ admissible}\}$ is a Z_2 basis for Λ. Let $\{\lambda^I \mid I \text{ admissible}\}$ be the dual basis of $\Lambda^* = \mathrm{Hom}(\Lambda, Z_2)$.

Let U_l be the vector space over Z_2 with basis the symbols $\lambda^I u_\omega$ where I is admissible, $2^i - 1 \notin \omega$, $l(I) = l$ and $2(t(I)+1) > n - |u_\omega|$. Grade U_l by $|\lambda^I u_\omega| = |\lambda^I| + |u_\omega|$. Let $d: A \otimes U_l \to A \otimes U_{l-1}$ be the A linear map defined by

$$d(1 \otimes \lambda^I u_\omega) = \sum \lambda^I(\lambda_j \lambda_J)\chi(Sq^j) \otimes \lambda^J u_\omega$$

where the sum ranges over all j and admissible J. Note by 2.2, if $\lambda^I(\lambda_j\lambda_J) \neq 0$, $t(J) \geqq t(I)$ and hence d is well defined. Let $\eta: A \otimes U_0 \to H^*(MO)$ be given by $\eta(a \otimes \lambda^{()} u_\omega) = a\Phi(u_\omega)$.

PROPOSITION 2.3. *The following sequence is exact:*

$$\longrightarrow A \otimes U_l \xrightarrow{d} A \otimes U_{l-1} \longrightarrow \cdots \longrightarrow A \otimes U_0$$

and

$$\Phi(I_n) = \eta(\text{image}\,(d: A \otimes U_1 \to A \otimes U_0))$$

We prove 2.3 in §3.

For a graded vector space V over Z_2, let $K(V)$ denote the Eilenberg–MacLane spectrum such that $\pi_*(K(V)) = V^*$ and $H^*(K(V)) = A \otimes V$.

PROPOSITION 2.4. *There is a sequence of Ω-spectra X_l, $l = 0, 1, 2, \ldots$ and maps $\alpha_l: X_{l-1} \to K(U_l)$ of degree $+1$ such that*

(i) $X_0 = K(U_0)$

(ii) *X_l is the fibration over X_{l-1} induced by α_l from the contractible fibration over $K(U_l)$.*

(iii) *If $i: K(U_l) \to X_l$ is the inclusion of the fibre of $X_l \to X_{l-1}$, $(\alpha_{l+1} i)^* = d: A \otimes U_{l+1} \to A \otimes U_l$.*

(iv) *If M is a smooth n-manifold, ν is its normal bundle, $g: MO \to K(U_0)$ realizes η and $h: T(\nu) \to MO$ comes from the classifying map of ν, then any lifting of $gh: T(\nu) \to X_0$ to X_{l-1} lifts to X_l.*

Since the X_l's are constructed from an acyclic complex,

$$\lim H^*(X_l) \approx \mathrm{Coker}\,(d: A \otimes U_1 \to A \otimes U_0) \approx H^*(MO)/\Phi(I_n).$$

To construct BO/I_n we essentially construct a tower of spaces

$$\to B_l \to B_{l-1} \to \cdots \to B_0 = BO$$

with fibres Eilenberg-MacLane spaces, such that if $T_l = T(\zeta_l)$ where $\zeta_l \to B_l$ is the pull back of the universal bundle over BO, then $T_l = X_l$ in dimensions $\leqq n$. We can then, more or less, define $BO/I_n = \lim B_l$.

We recall how the cohomology of a Thom space of a vector bundle changes, in a stable range, when a cohomology class in the base is killed. Suppose $g: B \to BO$ is a map such that $g_*: \pi_q(B) \approx \pi_q(BO)$ for $2q \leqq n$, V is a graded vector space with $V_q = 0$ for $2q \leqq n$ and $p: B' \to B$ is the fibration induced by a map $\gamma: B \to K(V)_1$ ($K(V) = \{K(V)_q\}$). Let $T = T(g^*\zeta)$ and $T' = T(p^*g^*\zeta)$. Viewing $B' \subset B$ as the fibre of γ, γ factors as $B \xrightarrow{i} B/B' \xrightarrow{\gamma'} K(V)_1$. Let

$$\Psi: (A(BO) \otimes V)^q \to H^{q+1}(T/T')$$

be given by $\Psi(a \circ u \otimes v) = a(u\Phi((\gamma')^*(v_1)))$ where $v_1 \in H^*(K(V)_1)$ is the element corresponding to $v \in V$ and Φ is the relative Thom isomorphism. In §6 we show that Ψ is an isomorphism for $q \leqq n$. (An equivalent form of this was proved in [1].) Combining this with the exact sequence of the pair (T, T') we obtain an exact sequence,

$$\to H^q(T) \to H^q(T') \to (A(BO) \otimes V)^q \to H^{q+1}(T) \to$$

for $q \leqq n$.

The cohomology of X_l and X_{l-1} are related by the Serre exact sequence,

$$\to H^q(X_{l-1}) \to H^q(X_l) \to (A \otimes U_l)^q \to H^{q+1}(X_{l-1}) \to.$$

Thus if we have constructed B_{l-1} such that $T_{l-1} = X_{l-1}$ in dimensions $\leqq n$ and we wish to construct B_l, we should take $B = B_{l-1}$ in the above and choose V_l so that $A(BO) \otimes V_l = A \otimes U_l$ as A modules. Our main algebraic result asserts that this is possible. Let

$$V_l = \{\lambda^I u_\omega \in U_l \mid \omega_r = \{\ \} \quad \text{for} \quad r \geqq l\}$$

PROPOSITION 2.5. *There are A linear isomorphisms $\theta: A \otimes U_l \to A(BO) \otimes V_l$ and $A(BO)$ linear maps $d: A(BO) \otimes V_l \to A(BO) \otimes V_{l-1}$, $l > 1$ and $d: A(BO) \otimes V_1 \to H^*(MO)$ such that the following diagram is commutative:*

$$\begin{tikzcd}
\longrightarrow A \otimes U_l \arrow[r, "d"] \arrow[d, "\theta"] & A \otimes U_{l-1} \arrow[r] \arrow[d, "\theta"] & \cdots \arrow[r] & A \otimes U_0 \arrow[d, "r"] \\
\longrightarrow A(BO) \otimes V_l \arrow[r, "d"] & A(BO) \otimes V_{l-1} \arrow[r] & \cdots \arrow[r] & H^*(MO).
\end{tikzcd}$$

Furthermore, if $u \in V_l \subset U_l$, *then* $\theta(1\otimes u)=1\otimes u$.

The construction of spaces B_l can now be made, modulo technical problems, using 2.5. Given B_{l-1} and f_{l-1}: $T_{l-1}\to X_{l-1}$, the k-invariant $\beta_l: B_{l-1}\to K(V_l)_1$ is defined by:

$$\Phi\beta_l^*(v_1)=f_{l-1}^*\alpha_l^*(v)$$

where $\alpha_l: X_{l-1}\to K(U_l)$ is the k-invariant for X_l, $v\in V$ and $v_1\in H^*(K(V)_1)$ corresponds to v. If M is an n-manifold and $h: \mathrm{M}\to BO$ classifies its normal bundle, 2.4(iv) shows that any lifting of h to B_{l-1} lifts to B_l. The $A(BO)$ linearity of d allows one (more or less) to construct $f_l: T_l\to X_l$. Actually, this straightforward procedure is marred by two technical details which we now describe.

Let $s=[n/2]$. To form B_1 from BO, one kills, among other things, the Wu class v_{s+1}, i.e. $d\lambda^s=\chi(Sq^{s+1})U=v_{s+1}U$, where the U is the Thom class. The map Ψ is zero on

$$\sum_{j>0}(Sq^j\circ v_{s+1-j})\otimes\lambda^s\in(A(BO)\otimes V_1)^{2s+1}$$

As a result, there is a class $x\in H^{2s+1}(X_1)$ which goes to zero in $H^{2s+1}(T_1)$. The class x is killed in going from X_1 to X_2. Hence if one were to follow the recipe given by 2.5, one would kill a class in B_1 which is already zero and thus produce a class in $H^{2s}(B_2)$ not coming from $H^{2s}(X_2)$. To avoid this, we omit a basis element from V_2. This same phenomena occurs in dimension $2s+2$ so we omit some more elements from V_2 and V_3. Namely, let $\bar V_l\subset V_l$ be spanned by $\lambda^I u_\omega\in V_l$ except $\lambda^{0,0}w_s^2$, $\lambda^{0,-1}w_{s+1}^2$, $\lambda^{-1,-2}w_{s+2}^2$ and for s odd, $\lambda^{-1,-2,-4}w_1^4w_s^2$ ($w_s=u_{(1,1,\dots,1)}$).

In §3 we define a certain $A(BO)$ linear map

$$r: A(BO)\otimes V_l\to A(BO)\otimes\bar V_l \tag{2.6}$$

such that $r\,|\,A(BO)\otimes\bar V_l$ is the identity. We then use $r\theta$ in place of θ in our construction of B_l.

The second difficulty arises in the following fashion. Again suppose we have B_{l-1} and $f_{l-1}: T_{l-1}\to X_{l-1}$ and we construct B_l using $\bar V_l$ instead of V_l as above. Let $g_l: T_{l-1}/T_l\to K(U_l)$ be the map such that $g_l^*(u)=\Psi r\theta(u)$ for $u\in U_l$. In order to construct $f_l: T_l\to X_l$ we need commutativity of the diagram

$$\begin{tikzcd} T_{l-1} \arrow[r, "j"] \arrow[d, "f_{l-1}"'] & T_{l-1}/T_l \arrow[d, "g_l"] \\ X_{l-1} \arrow[r, "\alpha_l"] & K(U_l). \end{tikzcd}$$

We can only prove that this diagram commutes in dimensions $\leqq 2s+1$. To correct for this we relabel B_l above, B_l' and we form B_l from B_l' by killing the obstructions to commutativity as follows:

Define $\Delta = \Delta(f_{l-1}): U_l \to H^*(T_{l-1})$ by

$$\Delta(u) = f_{l-1}^* \alpha_l^* u - \sum x_i f_{l-1}^* \alpha_l^* u_i$$

where $r\theta(u) = \sum x_i u_i$, $x_i \in A(BO)$, $u_i \in \bar{V}_l$. Then

$$j^* g_l^*(u) = j^* \Psi r\theta(u) = j^* \Psi\Big(\sum x_i u_{i_1}\Big) = \sum x_i j^* \Phi((\beta_l')^*(u_{i_1}))$$

$$= \sum x_i \Phi(\beta_l^*(u_i)) = \sum x_i f_{l-1}^* \alpha_l^*(u_i) = \Delta(u) + f_{l-1}^* \alpha_l^*(u)$$

Thus Δ is the deviation from commutativity of our diagram above. Let $W_l = U_l/\ker \Delta$. We kill $\Phi^{-1}(\Delta(W))$ in B_l' to form B_l.

To recapitulate, we inductively construct a sequence of spaces B_l, stable vector bundles ζ_l over B_l and maps $f_l: T_l = T(\zeta_l) \to X_l$ such that $\Delta(f_l) = 0$. We take $B_0 = BO$, $\zeta_0 = \zeta$ the universal bundle and f_0 the map such that $f_0^*(u_\omega) = \Phi(u_\omega)$ for $u_\omega \in U_0$. ($X_0 = K(U_0)$.) Referring to 2.5, $f_0^* = \eta$, $\alpha_1^* = d$ and $\Delta(f_0) = \eta d - d\theta = 0$. Suppose B_{l-1}, ζ_{l-1} and f_{l-1} have been defined and $\Delta(f_{l-1}) = 0$. Let $p': B_l' \to B_{l-1}$ be the fibration induced by $\beta_l: B_{l-1} \to K(\bar{V}_l)_1$ where β_l is defined by

$$\Phi(\beta_l^*(v_1)) = f_{l-1}^* \alpha_l^*(v)$$

for $v \in \bar{V}_l \subset U_l$ and $v_1 \in H^*(K(\bar{V}_l)_1)$ the element corresponding to v. Let $\zeta_l' = (p')^* \zeta_{l-1}$ and $T_l' = T(\zeta_l')$.

Viewing $B_l' \subset B_{l-1}$ as the fibre of β_l, β_l factors through β_l'. $B_{l-1}/B_l' \to K(\bar{V}_l)_1$. Let $\Psi: A(BO) \otimes \bar{V}_l \to H^*(T_{l-1}/T_l')$ be the $A(BO)$ linear map such that $\Psi(v) = \Phi((\beta_l')^*(v_1))$ for $v \in \bar{V}_l$. Let θ be as in 2.5, r as in 2.6, and let $g_l': T_{l-1}/T_l' \to K(U_l)$ be defined by $(g_l')^*(u) = \Psi r\theta(u)$. Since $\Delta(f_{l-1}) = 0$, there is a map f_l' making a commutative diagram

$$\begin{tikzcd}
T_{l-1}/T_l' \arrow[r] \arrow[d, "g_l'"] & T_l' \arrow[r] \arrow[d, "f_l'"] & T_{l-1} \arrow[r] \arrow[d, "f_{l-1}"] & T_{l-1}/T_l' \arrow[d, "g_l'"] \\
K(U_l) \arrow[r, "i"] & X_l \arrow[r] & X_{l-1} \arrow[r, "\alpha_l"] & K(U_l).
\end{tikzcd}$$

Let $\Delta(f_l'): U_{l+1} \to H^*(T_l)$ be given by $\Delta(f_l')(u) = (f_l')^* \alpha_{l+1}^* u + \sum x_i (f')^* \alpha_{l+1}^* u_i$ where $r\theta u = \sum x_i u_i$. Let $W_{l+1} = U_{l+1}/\ker \Delta(f_l')$ and let $p: B_l \to B_l'$ be the fibration induced by $\gamma_l : B_l' \to K(W_{l+1})_1$ where $\Phi(\gamma_l^* u_1) = \Delta(f_l')(u)$ for $u \in W_{l+1}$. Finally let $\zeta_l = p^*\zeta_l'$ and $f_l = f_l' T(p)$. Then $\Delta(f_l) = T(p)^* \Delta(f_l') = 0$ and the inductive step is complete.

In §5 we prove:

LEMMA 2.7. *If $l \geqq 3$ and $q \leqq n$, $f_l^* : H^q(X_l) \approx H^q(T(\zeta_l))$. Furthermore, if* M *is a smooth n-manifold and $h : \mathrm{M} \to B_0 = BO$ classifies its normal bundle, then any lifting of h to B_{l-1} lifts to B_l.*

We next examine $H^*(B_l)$ for l large.

LEMMA 2.8. *If $l \geqq n$, $V_l^q = U_l^q = 0$ for $q < n-1$, $W_l^q = 0$ for $q \leqq n$ and*

$V_l^{n-1} = U_l^{n-1} = \{\lambda^{(0,0,\dots,0)} u_\omega \mid u_\omega \in U_0^{n-1}\}$. *Furthermore,*

$$\Phi(\beta_l^*(\lambda^{(0,\dots,0)} u_\omega)) = \delta_l \tilde{u}_\omega$$

$\tilde{u}_\omega \in H^*(T_{l-1}; Z_{2l})$, $u_\omega U \in H^*(T_{l-1})$ is the mod *two reduction of $\tilde{u}_\omega$ and δ_l is the Bockstein associated with $Z_2 \to Z_{2l+1} \to Z_{2l}$.*

Thus for $l \geqq n$,

$$H^q(B_l) \approx H^q(BO)/I_n^q \qquad q < n$$

$$H^n(B_l)/\Phi^{-1}\{\delta_{l+1}\tilde{u}_\omega\} \approx H^n(BO)/I_n^n$$

We form B_∞ from B_l, $l \geqq n$, by killing classes $\Phi^{-1}(\delta^{l+1}\, \tilde{u}_\omega) \in H^{n+1}(B_l; Z_\tau)$ where Z_τ denotes twisted integer coefficients, twisted by w_1, $\Phi : H^*(B_l; Z_\tau) \approx H^*(T(\zeta_l); Z)$ is the Thom isomorphism and δ^l is the Bockstein associated with $Z \to Z \to Z_{2^l}$. Let $\tilde{B}_l$ be the two sheeted cover of $\tilde{B}_l$ defined by w_1. The classes $\Phi^{-1}(\delta^{l+1}\, \tilde{u}_\omega)$ may be represented by Z_2- equivariant maps $x_\omega : \tilde{B}_l \to K(Z, n)$ where $K(Z, n)$ has the action defined by the nontrivial action of Z_2 on Z. Let $\tilde{B}_\infty$ be the fibration over $\tilde{B}_l$ induced by

$$x = \prod x_\omega : \tilde{B}_l \to \prod K(Z, n)$$

Since x is Z_2-equivariant, Z_2 acts freely on $\tilde{B}_\infty$. Let $B_\infty = \tilde{B}_\infty / Z_2$. The map $B_\infty = \tilde{B}_\infty/Z_2 \to \tilde{B}_l/Z_2 = B_l$ has fibre $\Pi K(Z, n)$. With Z_2 coefficients, $\pi_1(B_l)$ acts trivially on the cohomology of the fibre. The Serre spectral sequences, with Z_2 coefficients has its usual, nonlocal coefficient form and the usual argument shows that in dimensions $\leqq n$,

$$H^*(BO_\infty) = H^*(B_l)/\{\Phi^{-1}(\delta^{l+1}\tilde{u}_\omega)\}.$$

Thus for $q \leqq n$

$$0 \to I_n^q \to H^q(BO) \to H^q(B_\infty) \to 0$$

is exact. Also if M is an n-manifold and $h: \mathrm{M} \to B$ is covered by a bundle map $g: \nu \to \zeta_l$, $T(g)^*(\delta^{l+1}\tilde{u}_\omega) = \delta^{l+1}T(g^*)(\tilde{u}_l) = 0$ since the top homology class of $T(\nu)$ is spherical. Therefore, h lifts to B_∞.

Finally, assume B_∞ is a CW complex and let

$$BO/I_n = B_\infty^n \cup e_1^{n+1} \cup e_2^{n+1} \cdots e_m^{n+1}$$

where e_i^{n+1} is attached by $f_i \mid S^n$, $f_i: (D^{n+1}, S^n) \to (B^{n+1}, B^n)$ and $[f_i] \in \pi_{n+1}(B_\infty^{n+1}, B_\infty^n)$ give a Z_2-basis for the image of

$$\pi_{n+1}(B_\infty^{n+1}, B_\infty^n) \xrightarrow{\rho} H_{n+1}(B_\infty^{n+1}, B_\infty^n) \xrightarrow{\partial^*} H_n(B_\infty^n, B_\infty^{n-1})$$

The maps f_i give an extension of $B_\infty^n \subset B_\infty$, $f: BO/I_n \to B_\infty$ and

$$f^*: H^q(B_\infty) \approx H^q(BO/I_n) \quad \text{for} \quad q \leqq n$$
$$H^q(BO/I_n) = H^q(BO)/I_n = 0 \quad \text{for} \quad q > n$$

Also any map of an n-manifold into B_∞ is homotopic to a map factoring through f. The proof of Theorem 1 is thus complete, modulo the lemmas and propositions of this section.

§3. Proofs of 2.3, 2.5, and 2.6

Let Λ_l^k be the Z_2-subspace of Λ^* generated by λ^I with $l(I) = l$, $t(I) \geqq k$, and I admissible. Let

$$d: A \otimes \Lambda_l^k \to A \otimes \Lambda_{l-1}^k$$

be defined by

$$d(1 \otimes \lambda^I) = \sum \lambda^I(\lambda_j \lambda_J)\chi(Sq^{j+1}) \otimes \lambda^J \tag{3.1}$$

where the sum is over all j and admissible J. Proposition 2.3 follows from 2.1 and 3.2(ii) below:

PROPOSITION 3.2.

(i) $\{\lambda_I \mid I$ *admissible*$\}$ *is a* Z_2*-basis for* Λ.
(ii) *The following is exact:*

$$\longrightarrow A\otimes\Lambda_l^k \xrightarrow{d} A\otimes\Lambda_{l-1}^k \longrightarrow \cdots \longrightarrow A\otimes\Lambda_0^k \xrightarrow{\epsilon} A/A\{\chi(Sq^i) \mid i>k\}$$

where $\epsilon(a\otimes\lambda^{(\,)})=\{a\}$.

(iii) *If* I *and* J *are admissible*, $l(I)=l$, $l(J)=l-1$, *and* $I_l=(1,2,4,\dots,2^{l-1})$, *then* $\lambda^{I+rI_l}(\lambda_{j+r}\lambda_{J+2rI_{l-1}})=\lambda^I(\lambda_j\lambda_J)$.

Proof. For any sequence $T=(t_1,t_2,\dots,t_l)$ and integer r, let $h^r(\lambda_T)=\lambda_{T+rI_l}$. Extending linearly, h^r gives a well defined map $h^r:\Lambda\to\Lambda$ since for any element of Λ of the form $\alpha=\lambda_{I_1}\beta\lambda_{I_2}$ where β is a relation for Λ as in 2.2, $h^r(\alpha)$ also has this form. Since h^rh^{-r} is the identity, h^r is an isomorphism for all r. Furthermore, $h^r(\lambda_I)$ is admissible if and only if λ_I is admissible.

Let $\bar{\Lambda}\subset\Lambda$ be the subalgebra generated by $\lambda_0,\lambda_1,\lambda_2,\dots$. In [8] it is proved that $\{\lambda_I \mid I$ admissible$\}$ is a basis for $\bar{\Lambda}$. For any λ_I, $h^r(\lambda_I)\in\bar{\Lambda}$ for r sufficiently large. Thus $\{\lambda_I \mid I$ admissible$\}$ is a basis for Λ.

In [2], 3.2(ii) was proved for $k\geqq 0$. From 2.2 one sees that $\lambda_{-1}\lambda_{-1}=0$ and if $t(J)\geqq 0$, $\lambda_{-1}\lambda_J$ is a sum involving $\lambda_{J'}$'s with $t(J')>0$ and $\lambda_J\lambda_{-1}$. Suppose $J_1=(j_i,\dots,j_m)$, $J_2=(j_{m+1},\dots,j_l)$ and $J=(j_1,\dots,j_l)$ are admissible with J_1 or J_2 possibly the empty sequence ( ). Define $\lambda^{J_1}\lambda^{J_2}=\lambda^J$. Suppose $j_m\geqq 0$ and $j_{m+1}<-1$. Then 3.1 yields

$$d(\lambda^{J_1}\lambda^{-1}\lambda^{J_2})=(d\lambda^{J_1})\lambda^{-1}\lambda^{L_2}+\lambda^{J_1}\lambda^{J_2}$$

$$d(\lambda^{J_1}\lambda^{J_2})=(d\lambda^{J_1})\lambda^{J_2}.$$

Let

$$D(\lambda^{J_1}\lambda^{J_2})=\lambda^{J_1}\lambda^{-1}\lambda^{J_2},\ D(\lambda^{J_1}\lambda^{-1}\lambda^{J_2})=0.$$

Then for $k<0$, $D:A\otimes\Lambda_l^k\to A\otimes\Lambda_{l+1}^k$ satisfies $dD+Dd=\text{identity}$. Therefore 3.2(ii) holds for $k<0$.

Finally we prove 3.2(iii). Note that if I is admissible, $I+rI_l$ is admissible and if $(h^r)^*:\Lambda^*\to\Lambda^*$ is the dual of h^r, h^r, $(h^r)^*\lambda^I=\lambda^{I-rI_l}$. Therefore

$$\lambda^I(\lambda_j\lambda_J)=(h^r)^*(\lambda^{I+rI_l})(\lambda_j\lambda_J)$$
$$=\lambda^{I+rI_l}(h^r(\lambda_j\lambda_J))=\lambda^{I+rI_l}(\lambda_{j+r}\lambda_{J+2rI_{l-1}})$$

Proof of 2.5. Let $C_l=A\otimes U_l$, $D_l=A(BO)\otimes V_l$, $l>0$, and $D_0=H^*(MO)$. Denote $a\otimes u\in C_l$ by au and $a\circ v\otimes w\in D_l$, $l>0$, by $(a\circ v)w$. We filter C_l and D_l as follows: $F_q(C_l)$ is spanned by $a\lambda^I u_l$ with $|u_\omega|\leqq q$ and $F_q(D_l)$, $l>0$, is spanned by all $a\circ v\lambda^I u_l$ with $|u_\omega|+2^l|v|\leqq q$. $F_q(D_0)$ is spanned by all au_ω where $a\in A$, $u_\omega\in U_0=\{u_\omega \mid 2^i-1\notin\omega\}$ and $|u_\omega|\leqq q$.

The chain complex (C_l,d) is a direct sum of chain complexes of the form described in 3.2, indexed by the $u_\omega\in U_0$. Hence d is filtration preserving and:

(3.3) The following is exact.

$$\longrightarrow F_q(C_l)\xrightarrow{d}F_q(C_{l-1})\longrightarrow\cdots\longrightarrow F_q(C_0)$$

Using induction on l we define A linear maps $\theta:C_l\to D_l$ and $A(BO)$ linear maps $d:D_l\to D_{l-1}$ such that

(i) θ is an isomorphism and $\theta:C_0\to D_0$ is given by $\theta(a\otimes u_\omega)=a\Phi(u_\omega)\in H^*(MO)$, $u_\omega\in U_0$.

(ii) $d\theta=\theta d$

(iii) If $u\in V_l\subset U_l$, $\theta(u)=u$

(iv) $\theta(F_q(C_l))=F_q(D_l)$

(v) Suppose $\lambda^I u_\omega\in U_l$. Let α and β be the partitions

$$\alpha=\bigcup_{r<l}2^r\omega_r,\qquad \beta=\bigcup_{r\geqq l}2^{r-l}\omega_r$$

Note $u_\omega=u_\alpha u_\beta^{2^l}$. Then θ satisfies

$$\theta(\lambda^I u_\omega)=u_\beta\lambda^{I'}u_\alpha \mod F_{|u_\omega|-1}(D_l)$$

where $I'=I+|u_\beta|\,I_l$.

Note that Proposition 2.5 consists of statements (i), (ii), and (iii) above.

For $l=0$, θ is defined by (i) and $d=0$ on D_0.

Suppose θ and d have been defined on C_k and D_k $k<l$, and satisfy (i)−(v). Define $d=d_D:D_l\to D_{l-1}$ to be the $A(BO)$ linear map such that for $u\in V_l$,

$d_D(u)=\theta(d_C u)$. We next define $\theta: C_l \to D_l$. Suppose $\lambda^I u_\omega \in U_l$ and $u_\omega = u_\alpha u_\beta^{2^l}$ as in (v). If $u_\beta = 1$, $\lambda^I u_\omega \in V_l$ and we define $\theta(\lambda^I u_\omega)=\lambda^I u_\omega$. In this case (i)–(v) are satisfied. Suppose $u_\beta \neq 1$. Let

$$X=\theta(d(\lambda^I u_\omega))+u_\beta \theta(d\lambda^{I'} u_\alpha)$$

where $I'=I+|u_\beta|\, I_l$. By induction, $\theta d = d\theta$ on C_{l-1} and hence $\partial X=0$. We show that $X \in F_{p-1}(D_l)$ where $p=|u_\omega|$. Decompose u_α into $u_{\alpha_1}u_{\alpha_2}^{2^{l-1}}$ as in (v).

$$\begin{aligned}\theta(d\lambda^I u_\omega) &= \sum \lambda^I(\lambda_j \lambda_K)\chi(Sq^{j+1})\theta(\lambda^K u_\omega)\\ &= \sum \lambda^I(\lambda_j\lambda_K)(\chi(Sq^{j+1})\circ u_{\alpha_2}u_\beta^2)\lambda^{K'}u_{\alpha_1} \bmod F_{p-1}\end{aligned}$$

where $K'=K+|u_{\alpha_2}u_\beta^2|\, I_{l-1}$. On the other hand,

$$u_\beta\theta(d\lambda^{I'}u_\alpha)=\sum \lambda^{I'}(\lambda_j\lambda_J)u_\beta\chi(Sq^{j+1})\theta(\lambda^J u_\alpha)$$

In $A(BO)$,

$$u_\beta\chi(Sq^{j+1})=\chi(Sq^{j-q+1})\circ u_\beta^2+\sum_{k<q}\chi(Sq^{j-k+1})\circ Sq^k u_\beta$$

where $q=|u_\beta|$.

$$\theta(\lambda^J u_\alpha)=u_{\alpha_2}\lambda^{J'}u_{\alpha_1} \bmod F_{|u_\alpha|-1}$$

where $J'=J+|u_{\alpha_2}|\, I_{l-1}$. If $u\lambda^I v$ has filtration less than $|u_\alpha|-1$ and $k<q$, $Sq^k u_\beta u\lambda^I v$ has filtration less than $p=|u_\omega|$.

Hence

$$u_\beta\theta(d\lambda^{I'}u_\alpha)=\sum_{j,J}\lambda^{I'}(\lambda_j\lambda_J)(\chi(Sq^{j-q+1})\circ u_\alpha u_\beta^2)\lambda^{J'}u_{\alpha_1} \bmod F_{p-1}$$

In the above sum, replace j by $j+q$ and J by $K+2qI_{l-1}$. Then

$$u_\beta\theta(d\lambda^{I'}u_\alpha)=\sum_{j,K}\lambda^{I'}(\lambda_{j+q}\lambda_{K+2qI_{l-1}})\chi(Sq^{j+1})\circ u_{\alpha_2}u_\beta^2\lambda^{K'}u_{\alpha_1} \bmod F_{p-1}$$

where $K'=K+|u_{\alpha_2}u_\beta^2|\, I_{l-1}$. But $I'=I+qI_l$ and hence by 3.2(iii),

$$\lambda^{I'}(\lambda_{j+q}\lambda_{K+2qI_{l-1}})=\lambda^I(\lambda_j\lambda_K)$$

Hence $X \in F_{p-1}(D_l)$.

By (iv) there is a $Y \in F_{p-1}(C_{l-1})$ such that $\theta(Y) = X$ and by (i) and (ii), $dY = 0$. Hence for $l > 1$, by 3.3, there is a $Z \in F_{p-1}(C_l)$ such that $dZ = Y$. We verify that there is such a Z for $l = 1$ by showing that when $l = 1$, $X \in \Phi(I_n)$. In this case

$$\begin{aligned} X &= \chi(Sq^{i+1})\Phi(u_\alpha u_\beta^2) + u_\beta \chi(Sq^{i+q+1})\Phi(u_\alpha) \\ &= \sum_{j<q} \chi(Sq^{i+q+1-j})\Phi((Sq^j u_\beta)u_\alpha) \end{aligned}$$

where $2(i+1) > n-q$, $q = |u_\beta|$. But then, $2(i+q-j+1) > n - |(Sq^j u_\beta)u_\alpha|$ and hence $X \in \Phi(I_n)$.

We now define $\theta(\lambda^I u_\omega)$ by induction on $|u_\omega|$ = filtration degree of $\lambda^I u_\omega$. For $|u_\omega| = 0$, $\theta(\lambda^I 1) = \lambda^I 1$. If θ is defined on $F_{|u_\omega|-1}(C_l)$, let

$$\theta(\lambda^I u_\omega) = u_\beta \lambda^{I'} u_\alpha + \theta(Z)$$

where Z, α, β, and I' are as above. Then $d\theta(Z) = \theta(dZ) = \theta(Y) = X$ and

$$\begin{aligned} d\theta(\lambda^I u_\omega) &= d(u_\beta \lambda^{I'} u_\alpha) + d\theta(Z) \\ &= u_\beta \theta(d(\lambda^{I'} u_\alpha)) + X = \theta(d(\lambda^I u_\omega)) \end{aligned}$$

Note that elements of the form $u_\beta \lambda^{I'} u_\alpha$, as above, together with $F_{p-1}(D_l)$, span $F_p(D_l)$ over A. Thus $\theta: C_l \to D_l$ is an epimorphism. (It is at this point that we use λ^I where I has negative entries. For each $u_\beta \lambda^{I'} u_\alpha \in H^*(BO)V_l$ we need $\lambda^I u_\alpha u_{\beta_l}^{2^l} \in U_l$ such that $I' = I + |u_l|\, I_l$.) Elements of the form $\lambda^I u_\alpha u_\beta^{2^l}$ are an A basis for C_l and elements of the form $u_\beta \lambda^{I'} u_\alpha$ are an A basis for D_l. Hence $\theta: C_l \to D_l$ is an isomorphism and the proof of 2.5 is complete.

Proof of 2.6. Let $v_i \in H^*(BO)$ be the Wu classes, that is, $\Phi(v_i) = \chi(Sq^i)\Phi(1)$ where $\Phi: H^*(BO) \to H^*(MO)$ is the Thom isomorphism.

LEMMA 3.4.

$$v_i = \sum s_\omega$$

where the sum ranges over all ω with entries only of the form $2^j - 1$ and $|s_\omega| = i$.

Proof. We view $H^*(BO) \subset Z_2[t_1, t_2, \ldots]$, $|t_i| = 1$, and $t_1 t_2 \ldots$ as the Thom class. Let $Sq = Sq^0 + Sq^1 + \cdots$ and $v = v_0 + v_1 + \cdots$. Then

$$\chi(Sq)t_i = \sum t_i^{2^j}$$

and

$$v(t_1, t_2, \ldots)(t_1 t_2 \cdots) = \chi(Sq)(t_1 t_2 \cdots)$$
$$= \prod_i \Big(\sum_j t_i^{2j-1}\Big)(t_1 t_2 \cdots) = \Big(\sum_\omega s_\omega\Big)(t_1 t_2 \cdots)$$

where the sum ranges over ω with entries only of the form $2^j - 1$.

Let x_1 and $x_2 \in A(BO)$ be given by

$$x_1 = \sum_{j>0} Sq^j \circ v_{s+1-j}, \qquad x_2 = \sum Sq^j \circ v_{s+2-j}$$

Recall $s = [n/2]$ and n is the dimension of the manifolds we are considering. Let $y_i^1 \in D_1$ be defined by

$$y_1^1 = x_1 \lambda^s, \qquad y_2^1 = x_2 \lambda^s, \qquad y_3^1 = v_{s+1}\lambda^{s+1} + v_{s+2}\lambda^s + x_2\lambda^s$$

LEMMA 3.5. *There are elements $y_i^2 \in D_2$ such that $dy_i^2 = y_i^1$ and*

$$y_1^2 = \lambda^{0,0} v_s^2 \bmod F_{2s-1}$$
$$y_2^2 = \lambda^{0,-1} v_{s+1}^2 \bmod F_{2s+1}$$
$$y_3^2 = \lambda^{-1,-2} v_{s+2} \bmod F_{2s+3}$$

If s is odd, there is an element y_2^3 such that $y_2^3 = (Sq^1 + w_1) y_2^2$ and

$$y_2^3 = \lambda^{-1,-2,-4} w_1^4 v_{s+2}^2 \bmod F_{2s+7}$$

Proof. We first show that $dy_i^1 = 0$, $d: D_1 \to D_0 = H^*(MO)$. Let $U \in H^0(MO)$ be the Thom class.

$$dy_1^1 = x_1 d\lambda^s = \sum Sq^j (v_{s+1-j}\chi(Sq^{s+1})U) + v_{s+1}\chi(Sq^{s+1})U$$
$$= (Sq^{s+1} v_{s+1})U + v_{s+1}^2 U = 0$$
$$dy_2^1 = \sum Sq^j (v_{s+2-j}\chi(Sq^{s+1})U)$$
$$= \sum Sq^j (v_{s+1}\chi(Sq^{s+2-j})U) = (Sq^{s+2} v_{s+1})U = 0$$
$$dy_3^1 = v_{s+1}\chi(Sq^{s+2})U + v_{s+2}\chi(Sq^{s+1})U + dy_2^1 = 0$$

We next show that y_1^2 exists. In $A \otimes \Lambda^*$ one may easily calculate $d\lambda^{0,0} = Sq^1\lambda^0$.

Hence, by the arguments in the proof of 2.5,

$$\begin{aligned} d\lambda^{0,0}v_s^2 &= \theta(d\lambda^{0,0}v_s^2) = \theta(Sq^1\lambda^0 v_s^2) \\ &= Sq^1 \circ v_s\lambda^s \bmod F_{2s-1} \\ &= \sum_{j>0} Sq^j \circ v_{s+1-j}\lambda^s \bmod F_{2s-1} = y_1^1 \bmod F_{2s-1} \end{aligned}$$

Thus $u = d\lambda^{0,0}v_s^2 + y_1^1 \in F_{2s-1}$ and $du = 0$. Therefore there is a $z \in F_{2s-1}(D_2)$ such that $dz = u$. Let $y_1^2 = \lambda^{0,0}v_s^2 + z$. The existence of y_2^2, y_3^2, and y_3^3 are proven in an analogous fashion.

We now define $r: A(BO)\otimes V_l \to A(BO)\otimes \bar{V}_l$. For $l \neq 2$ and $l \neq 3$, s odd, $\bar{V}_l = V_l$ and r is the identity; $\bar{V}_l \subset V_l$ and $r \mid A(BO)\otimes \bar{V}_l$ is the identity. $\bar{V}_2$ is formed from V_2 by omitting the basis elements $\lambda^{0,0}w_s^2$, $\lambda^{0,-1}w_{s+1}^2$ and $\lambda^{-1,-2}w_{s+2}^2$. By 3.4, v_i involves $w_i = s_{(1,1,\ldots,1)}$ when v_i is expressed in the u_ω basis. Let

$$\begin{aligned} r(\lambda^{0,0}w_s^2) &= y_1^2 - \lambda^{0,0}w_s^2 \\ r(\lambda^{0,-1}w_{s+1}^2) &= y_2^2 - \lambda^{0,-1}w_{s+1}^2 \\ r(\lambda^{-1,-2}w_{s+2}^2) &= y_2^3 - \lambda^{-1,-2}w_{s+2}^2 \end{aligned}$$

We define r on $A(BO)\otimes V_3$ analogously. Then $r(y_i^2) = r(y_2^3) = 0$.

We conclude this section with an algebraic lemma about the y_j^i's. Let $L_l \subset A(BO)\otimes V_l$ be defined as follows: $L_l = 0$ for $l = 0$, $l = 3$ and s even, and $l > 3$.

$$L_1 = A(BO)(\{y_i^1\} + S_1)$$

where $S_1 = \{v_3 Sq^2\lambda^2\}$ when $s = 2$ and $S_1 = 0$ for $s \neq 2$.

$$L_2 = A(BO)(\{y_i^2\} + S)$$

where $S_2 = \{v_3\lambda^{1,2}\}$ when $s = 2$ and $S_2 = 0$, $s \neq 2$.

$$L_3 = A(BO)\{y_3^2\}$$

$(d(v_3\lambda^{1,2}) = v_3 Sq^2\lambda^2)$.

LEMMA 3.6. *$d(L_l) \subset L_{l-1}$, $r(L_l) = 0$ for $l > 1$ and the sequence*

$$\longrightarrow L_l \xrightarrow{d} L_{l-1} \longrightarrow \cdots \longrightarrow L_0$$

is exact at L_l^q for all l and $q \leq 2s+2$.

Proof. The first part of 3.6 is clear from the definition of L_l. One easily checks that if $x \in A(BO)$, $|x| \leqq 1$ and $d(xy_2^3)=0$, then $x=0$ and therefore $d: L_3^q \to L_2^{q+1}$ is an injection for $q \leqq 2s+2$. $d: L_2 \to L_1$ is clearly onto. To check exactness at L_2^q, $q \leqq 2s+2$ one must verify that if $y = x_1 y_1^1 + x_2 y_2^1 + x_3 y_3^1 + x_4 v_3 Sq^2 \lambda^2 = 0$, $x_i \in A(BO)$ and $|y| \leqq 2s+3$, then $x_1 = x_3 = x_4 = 0$ and $x_2 = 0$ or s is odd and $x_2 = Sq^1 + w_1$. This is a tedious but straightforward calculation, made somewhat simpler by the following observation. Let

$$F: A(BO) \otimes \{\lambda^s\} \to H^*(MO \wedge K(Z_2, N))$$

be given by

$$F(a \circ u\lambda^s) = a(u\chi(Sq^{s+1})U \otimes \iota_N)$$

Then

$$F(y_1^1) = v_{s+1} U \otimes \iota_N + U \otimes Sq^{s+1} \iota_N$$
$$F(y_2^1) = U \otimes Sq^{s+2} \iota_N$$
$$F(v_3 Sq^2 \lambda^2) = v_3^2 U \otimes Sq^2 \iota_N$$

We leave the details to the reader.

§4. Proofs of 2.4 and 2.8

Let $\{A \otimes \Lambda_l^k, d\}$ be the chain complex described in Proposition 3.2.

PROPOSITION 4.1. *For each integer k, there are Ω-spectra $Y_l = Y_l(k)$ and maps $\rho_l = \rho_l(k): Y_{l-1} \to K(\Lambda_l^k)$ of degree one, $l = 0, 1, 2, \ldots$ such that*

(i) $Y_0 = K(\Lambda_0^k)$. *Y_l is a fibration over Y_{l-1} induced by ρ_l from the contractible fibration over $K(\Lambda_l^k)$.*

(ii) *If $i: K(\Lambda_{l-1}^k) \to Y_{l-1}$ is the inclusion of the fibre,*

$$(\rho_l i)^* = d: A \otimes \Lambda_l^k \to A \otimes \Lambda_{l-1}^k$$

where d is as in 3.2.

(iii) *If* M *is a smooth, compact n-manifold and ν is its normal bundle, then*

$$[T(\nu), Y_l]_p \to [T(\nu), Y_{l-1}]_p$$

is an epimorphism for $p < 2k+2$.

(iv) *Suppose* $k=0$. *Let* $I(l,0)=(0,\ldots,0)$ *have length* l.

$$\rho_l^* \lambda^{I(l,0)} = \delta_l \tilde{\iota}$$

where $\iota \in H^0(Y_{l-1}; Z_{2^l})$, $\tilde{\iota}$ *reduced modulo two is the generator* $\iota \in H^0(Y_{l-1}) \approx Z_2$ *and* δ_l *is the Bockstein associated to* $Z_2 \to Z_{2^{l+1}} \to Z_{2^l}$.

Proof. For $k \geqq 0$, 4.1(i), (ii), and (iii) were proved in [5]. For $k<0$, $\{A \otimes \Lambda_l^k, d\}$ is a free acyclic resolution of the zero A module so that the existence of Y_l and ρ_l easily follow by induction on l. If M is as in (iii), $v: T(\nu) \to Y_{l-1}$ has degree p, $p<2k+2$ and $k<0$, then $|(\rho_l v)^*(\lambda^I)|>n$ and (iii) follows.

Finally we prove (iv). The formula for d in 3.1 shows that $d\lambda^{I(l,0)} = Sq^1 \lambda^{I(l-1,0)}$ The complex,

$$\longrightarrow A \otimes \{\lambda^{I(l,0)}\} \xrightarrow{d} A \otimes \{\lambda^{I(l-1,0)}\} \longrightarrow \cdots A \otimes \{\lambda^{I(0,0)}\}$$

is realized by the tower

$$\to K(Z_{2^l}) \to K(Z_{2^{l-1}}) \to \cdots \to K(Z_2)$$

with k-invariants, $\delta_l : K(Z_{2^l}) \to K(Z_2)$. Except for $\lambda^{I(l,0)}$, the generators of Λ_l^0 have dimension >0 and hence kill classes of dimension >1. Thus $Y_l = K(Z_{2^{l+1}})$ in dimensions $\leqq 1$. Therefore (iv) holds.

Proof of 2.4: We wish to realize the complex $\{A \otimes U_l, d\}$ by a tower of spectra, X_l. Let $Y_l(k)$ and $\rho_l(k)$ be as in 4.1. For a spectrum Z, let SZ denote the shift suspension, i.e., $(SZ)_q = Z_{q+1}$. Define X_l and $\alpha_l : X_{l-1} \to K(Y_l)$ by

$$X_l = \prod_{u_\omega \in U_0} S^{|u_\omega|} Y_l([(n-|u_\omega|)/2])$$

$$\alpha_l = \prod S^{|u_\omega|} \rho_l([(n-|u_\omega|)/2])$$

The map α_l takes X_{l-1} into $K(U_l)$ since

$$\prod S^k K(\Lambda_l^k) = K(U_l)$$

where k ranges over $[(n-|u_\omega|/2]$, $|u_\omega| \in U_0$. Proposition 2.4 now follows directly from 4.1.

Proof of 2.8: Using induction on l, one easily proves that if I is admissible and $l=l(I)$,

$$|\lambda^I|\geqq 2t(I)\left(1-\frac{1}{2^l}\right)$$

Suppose $l\geqq n$ and $\lambda^I u_\omega\in U_l$. Then $2(t(I)+1)>n-|u_\omega|$. Therefore

$$|\lambda^I u_\omega|\geqq 2t(I)\left(1-\frac{1}{2^l}\right)+|u_\omega|\geqq n-1-\frac{n-|u_\omega|-1}{2^l}>n-2$$

Also if $|u_\omega|>n-1$, $|\lambda^I u_\omega|>n-1$. If $|u_\omega|<n-1$, $t(I)\geqq 1$ and hence $|\lambda^I|\geqq l\geqq n$. Therefore $U_l^q=0$ for $q<n-1$ and $U_l^{n-1}=\{\lambda^{I(l,0)}u_\omega \mid u_\omega\in U_0^{n-1}\}$ since $\lambda^{I(l,0)}$ is the only λ^I with $t(I)\geq 0$ and $|\lambda^I|=0$. If $r>l$ and $\omega_r\neq\{\ \}$, $|u_\omega|\geq|u_{\omega_r}^{2^r}|\geq 2^r>n$. Hence $V_l^q=U_l^q$ for $q\leqq n-1$.

By the definition of $\beta_l:B_{l-1}\to K(V_l)$,

$$\Phi(\beta_l^*(\lambda^{I(l,0)}u_\omega))=f_{l-1}^*\alpha_l^*(\lambda^{I(l,0)}u_\omega)$$

By 4.1(iv) $\alpha_l^*(\lambda^{I(l,0)}u_\omega)=\delta_l\bar\iota$ where $\bar\iota\in H^*(X_{l-1};Z_{2^l})$ comes from the factor of X_{l-1}, $Y([n-|u_\omega|/2])$. Since the diagram

$$\begin{array}{ccc} T_{l-1} & \xrightarrow{f_{l-1}} & X_{l-1} \\ \downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle p_2} \\ T_0 & \xrightarrow{f_0} & X_0 \end{array}$$

commutes, $\bar u=f_{0-1}^*\bar\iota$ reduced modulo two is $p_1^*f_0^*u_\omega=p_1^*u_\omega U_0=u_\omega U_{l-1}$, where U_l is the Thom class of T_l and the proof of 2.8 is complete.

§5. Proof of 2.7

If G_1 and G_2 are graded groups and $h:G_1\to G_2$ is a homomorphism of degree i, we will say that h is k connected if $h:G_1^q\to G_2^{q+i}$ is an epimorphism for $q<k$ and a monomorphism if $q\leqq k$. We will say that a sequence of graded groups and homorphisms,

$$\cdots\to G_l\to G_{l-1}\to\cdots$$

is k-exact if

$$G_{l+1}^{q-i} \to G_l^q \to G_{l-1}^{q+j}$$

is exact for all l and $q \leqq k$.

In §3 we constructed isomorphisms $\theta : A \otimes U_l \to A(BO) \otimes V_l$ and a subcomplex $\{L_l, d\} \subset \{A(BO) \otimes V_l, d\}$ such that

$$\longrightarrow L_l \xrightarrow{d} L_{l-1} \xrightarrow{d} \cdots \longrightarrow L_0 = 0$$

is $2s+2$ exact, $s = [n/2]$. In §4 we constructed a tower of fibrations $\to X_l \to X_{l-1} \to$ with k-invariants $\alpha_l : X_{l-1} \to K(U_l)$ associated to the complex $\{A \otimes U_l, d\}$. Let

$$\bar{H}^*(K(U_l)) = H^*(K(U_l))/\theta^{-1}(L_l)$$
$$\bar{H}^*(X_l) = H^*(X_l)/\alpha_{l-1}^*\theta^{-1}(L_{l-1})$$

LEMMA 5.1: *The maps*

$$K(U_l) \xrightarrow{i} X_l \xrightarrow{p} X_{l-1} \xrightarrow{\alpha_l} K(U_l)$$

induce a $2s+2$*-exact sequence*

$$\to \bar{H}^*(K(U)) \to \bar{H}^*(X_{l-1}) \to \bar{H}^*(X_l) \to$$

Proof: Let E_l be the kernel of

$$H^*(X_l) \to \lim_{k \to \infty} H^*(X_k)$$

Then $H^*(X_l) \approx H^*(\mathrm{MO})/\Phi(I_n) \oplus E_l$ and E_l and $A \otimes U_l$ are related by the diagram

$$\begin{tikzcd}[column sep=small]
{} \arrow[rr] & & A \otimes U_l \arrow[rrrr, "d"] \arrow[drr, "\bar{\alpha}_l"] & & & & A \otimes U_{l-1} \arrow[rrrr] \arrow[drr] & & & & A \otimes U_{l-2} \arrow[rr] & & {} \\
& & & & E_{l-1} \arrow[urr, "\bar{v}_2"] \arrow[dr] & & & & E_{l-2} \arrow[urr] \arrow[dr] & & & & \\
& & & 0 \arrow[ur] & & 0 & & 0 \arrow[ur] & & 0 & & &
\end{tikzcd}$$

where the $\bar{\alpha}_l$ and $\bar{i}_l$ are defined by α_l^* and i_l^* and each pair of composable arrows is exact. Dividing $A\otimes U_l$ and E_{l-1} by $\theta^{-1}(L_l)$ and $\bar{\alpha}_l\theta^{-1}(L_{l-1})$, respectively, produces the same type of diagram with exactness replaced by $2s+2$-exactness. The desired result then follows.

In §2 we defined maps

$$g_l': K(U_l) \to T_{l-1}/T_l'$$

In §6 we prove:

LEMMA 5.2. *The map* g_l' *induces a* $2s+2$*-connected map*

$$F_l: \bar{H}^*(K(U_l)) \to H^*(T_{l-1}/T_l')$$

for $l \geqq 1$.

Proof of 2.7: We first prove 2.7(ii). Suppose M is a smooth n-manifold, $h: \mathrm{M} \to B_0 = BO$ classifies ν, the normal bundle of M and $\tilde{h}: \mathrm{M} \to B_{l-1}$ is a lifting of h. Let $T(\tilde{h}): T(\nu) \to T_{l-1}$ denote the associated Thom space map. Then $f_{l-1}T(\tilde{h}): T(\nu) \to X_{l-1}$ is a lifting of $f_0T(h): T(\nu) \to X_0$ and hence by 2.4(iv), $f_{l-1}T(\tilde{h})$ lifts to X_l and therefore $\alpha_l f_{l-1}T(\tilde{h}) = 0$. Thus for $v \in \bar{V}_l$

$$\Phi h^*\beta_l^*(v_1) = T(\tilde{h})^*\Phi(\beta_l^*(v_1)) = T(\tilde{h})^*f_{l-1}^*\alpha_l^*(v) = 0$$

Thus $\beta_l\tilde{h} = 0$ and $\tilde{h}$ lifts to $h': \mathrm{M} \to B_l'$

If $u \in U_{l+1}$, $\bar{u} = \{u\} \in W_{l+1} = U_{l+1}/\ker \Delta$ and $\nu\theta(u) = \sum x_i u_i$, $x_i \in A(BO)$ and $u_i \in V_{l+1}$, then

$$\Phi((h')^*\gamma_l^*(\bar{u}_1)) = T(h')^*\Phi(\gamma_l^*\bar{u}_1) = T(h')\Delta(u).$$

Recall,

$$\Delta(u) = (f_l')^*\alpha_{l+1}^*u - \sum x_i(f_{l+1}')^*\alpha^*u_i$$

But $T(h')^*$ is $A(BO)$ linear and $\alpha_{l+1}f'T(h') = 0$ as above. Thus $T(h')^*\Delta(u) = 0$ and hence $\gamma_l h' = 0$. Therefore h' lifts to B_l and the proof of 2.7(ii) is complete. We note for further reference:

LEMMA 5.3: $T(h')^*\Delta(u) = 0$ *for* $u \in U_{l+1}$.

LEMMA 5.4. If $\delta^*: H^*(T_l') \to H^*(T_{l-1}/T_l')$, $\delta^*\Delta(u) = 0$ *for* $u \in U_{l+1}$.

Proof. Consider the commutative diagram:

$$\begin{tikzcd} H^*(X_l) \arrow[r, "i^*"] \arrow[d, "f_l'"] & H^*(K(U_l)) \arrow[d, "(g_l')"] \\ H^*(T_l') \arrow[r, "\alpha^*"] & H^*(T_{l-1}/T_l') \end{tikzcd}$$

Recall, g_0' realizes $\Psi r\theta$, $i^*\alpha_{l+1}^* = d$ and Ψ, r, and $d: A(BO)\otimes V_{l-1} \to A(BO)\otimes V_{l-1}$ are $A(BO)$ linear. Hence,

$$\begin{aligned} \delta^*\Delta(u) &= \delta^*((f_l')^*\alpha_{\alpha+1}^* u + \sum x_i(f')^*\alpha_{l+1}^* u_i) \\ &= (g_l')^* i^*\alpha_{l+1}^* u + \sum x_i (g_l')^* i^*\alpha_{l+1}^* u_i \\ &= \Psi r\theta du + \sum x_i \Psi r\theta du_i = \Psi r d\theta u + \sum \Psi r dx_i \theta(u_i) \end{aligned}$$

where $r\theta(u) = \sum x_i u_i$, $x_i \in A(BO)$ and $u_i \in V_{l+1}$. But for $v \in V_{l+1}$, $\theta(v) = v$. Thus

$$\sum x_i\theta(u_i) = \sum x_i u_i = r\theta u = \theta u + z$$

where $z \in L_{l+1}$. Furthermore $dz \in L_l$. Hence $\delta^*\Delta(u) = \Psi r dz = \Psi r\theta\theta^{-1}dz = (g')^*\theta^{-1}dz$.

But by 5.2, $\theta^{-1}(L_l)$ is the kernel of $(g_l')^*$.

We now prove that f_l induces a $2s+2$-connected map $\bar{f}_l : \bar{H}(X_l) \to H^*(T_l)$ by induction on $l \geqq 0$. We first show that $\bar{f}_l$ is well defined.

$$\bar{H}^*(X_l) = H^*(X_l)/\alpha_{l+1}^*(\theta^{-1}(L_{l+1}))$$

From the commutative diagram:

$$\begin{tikzcd} T_l \arrow[r, "j"] \arrow[d, "f_l"] & T_l/T_{l+1}' \arrow[d, "g_{l+1}'"] \\ X_l \arrow[r, "\alpha_{l+1}"] & K(U_{l+1}) \end{tikzcd}$$

we see that

$$f_l^*\alpha_{l+1}^*(\theta^{-1}(L_{l+1})) = j^*(g'_{+1})^*(\theta^{-1}(L_{l+1}))$$

By 5.2, $\theta^{-1}(L_{l+1})$ is in the kernel of $(g_{l+1}')^*$.

Since f_0^* is an isomorphism, $\bar{f}_0 = f_0^*$ and $\bar{f}_0$ is an isomorphism.

Suppose $\bar{f}_{l-1}$ is $2s+2$ connected. If $u \in U_{l+1}$, $\Delta(u) \in H^q(T_l')$ pulls back to $H^q(T_{l-1})$ since, by 5.4, $\delta^*\Delta(u) = 0$ and it pulls back to $H^q(X_{l-1})$ if $q < 2s+2$, that is, if $|u| < 2s+1$, $\Delta(u) = (f_l')^* p^* x$ where $p: X_l \to X_{l-1}$. But since the X_l's are constructed from an acyclic complex, image $p^* =$ image $(H^*(X_0) \to H^*(X_l))$. Therefore image $(f_l')^* p^* =$ image $(H^*(T_0) \to H^*(T_l')) = H^*(\mathrm{MO})/\Phi(I_n)$. But by 5.3, $\Delta(u)$ is zero on all n-manifolds. Hence $\Delta(u) = 0$ and we have shown that $W_{l+1}^q = (U_{l+1}/\ker \Delta)^q = 0$ for $q < 2s+1$. Therefore $H^q(B_l') \to H^q(B_l)$ is an isomorphism for $q \leqq 2s+2$ since B_l is a fibration over B_{l-1}' induced by $\gamma_l : B_l' \to K(W_{l+1})_1$. Then $H^q(T_l'/T_l) = H^q(B_l', B_l) = 0$ for $q < 2s+2$ and hence

$$H^*(T_{l-1}/T_l') \to H^*(T_{l-1}/T_l)$$

is $(2s+2)$-connected. Let g_l be the composition

$$T_{l-1}/T_l \longrightarrow T_{l-1}/T_l' \xrightarrow{g_l} K(U_l)$$

and let $\bar{g}_l : \bar{H}^*(K(U_l)) \to H^*(T_{l-1}/T_l)$ be induced by g_l. Then $\bar{g}_l$ is $(2s+2)$-connected by 5.2. Consider the commutative diagram:

$$\begin{tikzcd}
\arrow[r] & \bar{H}^*(K(U_l)) \arrow[r] \arrow[d, "\bar{g}_l"] & \bar{H}^*(X_{l-1}) \arrow[r] \arrow[d, "\bar{f}_{l-1}"] & \bar{H}^*(X_l) \arrow[r] \arrow[d, "\bar{f}\sim"] & \bar{H}^*(K(U)) \arrow[r] \arrow[d] & {} \\
\arrow[r] & H^*(T_{l-1}/T_l) \arrow[r] & H^*(T_{l-1}) \arrow[r] & H^*(T_l) \arrow[r] & H^*(T_{l-1}/T_l) \arrow[r] & {}
\end{tikzcd}$$

A five lemma argument and the fact that $\bar{f}_{l-1}$ and $\bar{g}_l$ are $(2s+2)$-connected shows that $\bar{f}_l$ is $2s+2$-connected.

Since $L_l = 0$ for $l > 3$, $\bar{H}^*(X_l) = H^*(X_l)$ for $l \geqq 3$ and therefore $f_l^* : H^q(X_l) \to H^q(T_l)$ is an isomorphism for $q \leqq n < 2s+2$. This completes the proof of 2.7.

§6. Proof of 5.2

LEMMA 6.1.

$$H^q(B_{l-1}) \to H^q(B_l')$$

is an isomorphism for $l > 1$ *and* $q \leqq s+1$. *For* $l = 1$ *it is an epimorphism for* $q \leqq s+1$ *and* v_{s+1}, $w_1 v_{s+1}$, $Sq^1 v_{s+1}$ *and* v_{s+2} *generate the kernel for* $q \leqq s+2$.

Proof. As we saw in the proof of 2.8, if $\lambda^I u_\omega \in V_l$, $|\lambda^I u_\omega| \geqq (n-1)-(n-|u_\omega|-1)/2^l$. Hence the lowest dimensional element in V_l is of the form λ^I with $t(I)=s$. For such an I, $|\lambda^I| \geqq s+2$ except for $l=1$ or $l=2$ and $s=1$ and 2. The space B_l' is a fibration over B_{l-1} induced by $\beta_l : B_{l-1} \to K(V_l)_1$ and for $l>1$, $K(V_l)_1$ is $s+2$ connected except when $l=2$ and $s=1$ or 2. For $s=1$ or 2, the lowest dimensional elements in V_2 are $\lambda^{1,1}$ and $\lambda^{1,2}$ respectively; $d\lambda^{1,1} \neq 0$ and $d\lambda^{1,2} \neq 0$ so these elements kill nonzero classes in B_1. Thus for $l>1$, $H^q(B_{l-1}) \approx H^q(B_l')$ for $q \leqq s+1$.

Suppose $l=1$. From 3.1 one sees that $d\lambda^i = \chi(Sq^{i+1})U = \Phi(v_{i+1})$ where U is the Thom class and v_{i+1} is the Wu class. Hence $\beta_1 : B_0 \to K(V_1)_1$ takes λ^i into v_{i+1}. One easily checks that $V_1^q = 0$ for $q<s$, $V_1^s = \{\lambda^s\}$ and $V_1^{s+1} = \{\lambda^{s+1}\}$. The remainder of 6.1 now follows by a simple Serre spectral sequence argument.

Let $K_l = K(V_l)_1$. Viewing $\beta_l : B_{l-1} \to K_l$ as a fibre map with fibre B_l', consider the pair of fibrations p_1 and p_2:

$$\begin{tikzcd}
(B_{l-1}, B_l') \arrow[r, "c"] \arrow[dr, "p_1"'] & (B_{l-1} \times K_l, B_{l-1} \times \{*\}) \arrow[d, "p_2"] \\
& (K_l, *)
\end{tikzcd}$$

where p_1 is defined by β_l, p_2 is projection on the second factor and $c = id \times p$. Note c is a fibre preserving map so we may use it to compare the Serre spectral sequences of p_1 and p_2.

LEMMA 6.2. *For $l>1$, $c^* : H^q(B_{l-1} \times K_l, B_{l-1} \times \{*\}) \to H^q(B_{l-1}, B_l')$ is an isomorphism for $q \leqq 2s+3$. For $l=1$, c^* is an epimorphism for $q \leqq 2s+2$ and for $q \leqq 2s+3$ the kernel is generated by*

$$v_{s+1} \otimes \lambda_1^s + 1 \otimes (\lambda_1^s)^2$$
$$v_{s+1} \otimes Sq^1 \lambda_1^s + 1 \otimes \lambda_1^s Sq^1 \lambda_1^s$$
$$v_{s+1} \otimes \lambda_1^{s+1} + 1 \otimes \lambda_1^s \lambda_1^{s+1}$$
$$w_1 v_{s+1} \otimes \lambda_1^s + w_1 \otimes (\lambda_1^s)^2$$
$$Sq^1 v_{s+1} \otimes \lambda_1^s + 1 \otimes \lambda_1^s Sq^1 \lambda_1^s$$
$$v_{s+2} \otimes \lambda_1^s + 1 \otimes \lambda_1^s \lambda_1^{s+1}$$

Proof. Let $E_r^{p,q}$ and $\bar{E}_r^{p,q}$ denote the Serre spectral sequences for p_1 and p_2 respectively.

$$E_2^{p,q} = H^p(K_l, *) \otimes H^q(B_{l-1})$$
$$\bar{E}_2^{p,q} = H^p(K_l, *) \otimes H^q(B')$$

As we saw above, for $l>1$, K_l is $s+2$ connected and $H^q(B_{l-1})\approx H^q(B_l')$ for $q\leqq s+1$. Therefore c induces an isomorphism at the E_2 level for $p+q\leqq 2s+3$ and the differentials are trivial for p_2 because it is a product fibration. This proves 6.2 for $l>1$.

For $l=1$, 6.2 is true at the E_2 level with the first summands in the above list of elements as a basis for the kernel; the second summands are of lower filtration. The same is true at the E_∞ level, so to complete the proof, we must show that these elements are in the kernel of c^*.

Under the map $H^*(B_0, B_1')\to H^*(B_0)$, $c^*(1\otimes\lambda_1^s)$ goes to v_{s+1}. Hence

$$c^*(v_{s+1}\otimes\lambda_1^s+1\otimes(\lambda_1^s)^2)=v_{s+1}c^*(1\otimes\lambda_1^s)+c^*(1\otimes\lambda_1^s)^2=0$$

(If $j:X\subset(X,A)$ and $x\in H^*(X,A)$, $x^2=(j^*x)x$.) The same argument applies to the other five elements.

Let

$$\phi:(A(BO)\otimes\bar{V}_l)^q\to H^{q+1}(T_{l-1}\wedge K_l)$$

be defined by

$$\phi((a\otimes w)u)=a(wU\otimes u_1)$$

where U is the Thom class, $a\in A$, $w\in H^*(BO)$ and $u\in\bar{V}_l$.

LEMMA 6.3. *For $q\leq 2s+1$, ϕ is an epimorphism. For $q\leq 2s+2$ the kernel of ϕ is zero for $l>1$ and $(l,s)\neq(2,2)$, is $\{v_3\lambda^{1,2}\}$ for $(l,s)=(2,2)$ and is $\{(\sum Sq^i\circ v_{s+2-i})\lambda^s\}$ for $l=1$.*

Proof. Let $\mu,\mu':A(BO)\to A(BO)$ be defined by

$$\mu(a\circ w)=\sum a_i'\circ w\varsigma(a_i'')$$
$$\mu'(a\circ w)=\sum a_i'\circ w\chi(a_i'')$$

(Recall, wa is defined by $(wa/U=\chi(a)(wU)$.)
Where $a\to\sum a_i'\otimes a_i''$ in the diagonal in A. Then $\mu\mu'=\mu'\mu=$ identity and thus μ is a Z_2-isomorphism. Let $\phi'=\phi(\mu\otimes id)$. Then

$$\phi'((a\circ w)u)=\sum a_i'(\chi(a_i'')(wU)\otimes u_1)=wU\otimes au_1$$

Let λ^I be the lowest dimensional element in $\bar{V}_l$; $|\lambda^I|>s$ for $l=1$. The lowest dimensional element in $H^*(T_{l-1}\wedge K_l)$ not in the image of ϕ' is $U\otimes(\lambda_1^I \cup Sq^i\lambda_1^I)$, an element of dimension $\geqq 2s+3$. Hence ϕ is an epimorphism for $q<2s+2$. The lowest dimensional elements in the kernel of ϕ' are $1\circ v_{s+1}\lambda_1^I$ or $(Sq^m\circ 1)\lambda_1^I$ where $m=|\lambda_1^I|+1$. For $l>2$, $(l, s)\neq(2, 2)$, $\lambda_1^I>s+1$ and hence these elements occur in dimensions $>2s+3$. For $(l, s)=(2, 2)$, $\phi(v_3\lambda^{1,2})=\phi'(v_3\lambda^{1,2})=0$. For $l=1$

$$0=\phi'((Sq^{s+2}\circ 1)\lambda^s)=\phi\Big(\Big(\sum Sq^i\circ v_{s+2-i}\Big)\lambda^s\Big)$$

This proves the last part of 6.3.

Proof of 5.2: We must show that

$$(g_l')^*=\Psi r\theta:(A\otimes U_l)^q\to H^{q+1}(T_{l-1}/T_l')$$

is an epimorphism for $q\leqq 2s+2$ and $(L_l)^q$ is the kernel for $q\leqq 2s+2$. By 2.5, θ is an isomorphism. Let ϕ be the map in 6.3 and c the map in 6.2. Lifting c to the Thom space level we obtain a map

$$T(c):T_{l-1}/T_l'\to T_{l-1}\wedge K_l$$

Furthermore $\Psi=T(c)^*$. Thus by 6.2 and 6.3, Ψ is an epimorphism for $q\leq 2s+1$ and since r is an epimorphism, $(g_l')^*$ is an epimorphism for $q\leq 2s+1$. For $l>1$ and $(l, s)\neq(2, 2)$, $T(c)^*$ and ϕ are monomorphisms for $q\leq 2s+2$ and L_l^q is the kernel of r. When $(l, s)=(2, 2)$ $r(L_l)=\{v_3\lambda^{1,2}\}$. This completes the proof of 5.2 for $l>1$.

Suppose $l=1$. Then $r=$identity. We wish to show that $L_1=\phi^{-1}(\ker T(c)^*)$. In 6.2 a basis for $\ker c^*$ was given for $q\leq 2s+2$. Since image $\phi=$image ϕ' cannot involve cup products (except squares) in $H^*(K_l)$, the above basis shows that the following is a basis for image $\phi\cap\ker T(c)^*$:

$$v_{s+1}U\otimes\lambda_1^s+U\otimes Sq^{s+1}\lambda_1^s$$

$$w_1v_{s+1}U\otimes\lambda_1^s+w_1U\otimes Sq^{s+1}\lambda_1^s$$

$$v_{s+1}U\otimes Sq^1\lambda_1^s+(Sq^1v_{s+1})U\otimes\lambda_1^s$$

$$v_{s+1}U\otimes\lambda_1^{s+2}+v_{s+2}U\otimes\lambda_1^s$$

Thus a basis for $\phi^{-1}(\ker c^*)$ is ϕ^{-1} of these elements and $(\sum Sq^i\circ v_{s+2-i})\lambda^s$ from the kernel of ϕ. A simple calculation shows that these elements form a basis for L_1^q, $q\leq 2s+2$, completing the proof of 5.2.

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