Classification of simple knots by Levine pairings.

Autor(en): **Kojima, Sadayoshi**

Objekttyp: **Article**

Zeitschrift: **Commentarii Mathematici Helvetici**

Band (Jahr): **54 (1979)**

PDF erstellt am: **19.05.2024**

Persistenter Link: https://doi.org/10.5169/seals-41583

Comment. Math. Helvetici **54** (1979) 356–367 Birkhäuser Verlag, Basel

Classification of simple knots by Levine pairings

by SADAYOSHI KOJIMA

§1. Introduction

An *n-knot* will be a smooth oriented submanifold K of S^{n+2}, where K is homeomorphic to S^n. By the Alexander duality theorem, the complement $X = S^{n+2} - K$ is a homology circle. Abelianization defines an epimorphism $\varepsilon : \pi_1(X) \to \mathbf{Z}$, and the covering space $\tilde{X} \to X$ associated to Ker $\varepsilon \subset \pi_1(X)$ is the universal abelian covering of X. The group of covering translations of $\tilde{X}$ is infinite cyclic, generated by t, and this defines a unique module structure on $H_q(\tilde{X} : \mathbf{Z})$ over the ring $\Lambda = \mathbf{Z}[t, t^{-1}]$. We will use the notation $A_q = H_q(\tilde{X})$ and refer to it as the q-th *Alexander module* of K.

An n-knot K is *simple* if $\tilde{X}$ is $[(n-1)/2]$-connected. In this paper, we shall study a classification of some simple knots. If $n = 2q-1$, there is a linking form $\langle , \rangle : A_q \times A_q — Q(\Lambda)/\Lambda$ satisfying the $(-1)^{q+1}$-Hermitian property, where $Q(\Lambda)$ is the quotient field of Λ. Moreover the isotopy class of the knot K is completely determined by $(A_q, \langle , \rangle)$ when K is simple, [K1], [L2], [T]. In case $n = 2q$, J. Levine has defined a more obscure linking form $[,]$ on the $\mathbf{Z}$-torsion submodule T_q of A_q with values in $\mathbf{Q}/\mathbf{Z}$, satisfying the $(-1)^{q+1}$-symmetric property. A simple $2q$-knot K is *odd-finite* if $\pi_q(X)$ is 2-torsion free and finite. The main result is

THEOREM 1. *If K_0, K_1 are odd finite simple $2q$-knots with isometric Levine pairings and $q \geq 4$, then K_0 is isotopic to K_1.*

Remark 1. A consequence of [B&L] is that a simple $2q$-knot is fibered if and only if $\pi_q(X)$ is finitely generated. In particular, any finite simple knot is fibered.

Remark 2. Some results are known about the classification of even-dimensional simple knots, [K2], [Ko]. They say that the isotopy class of a knot is in general not determined by the pairing structure on the Alexander module if $\pi_q(X)$ is infinite.

In §4, we shall state some geometrical properties of knots in case that $\pi_q(X)$ is finitely generated and 2-torsion free. Classification in Theorem 2 is established by summing up Theorem 1 and [Ko].

The author would like to express his hearty thanks to Professor J. Levine for introducing him to this problem with a letter. He also would like to thank Professor M. Kato and Professor C. Kearton for many useful comments.

§2. Levine pairings

We recall the definition of the Levine pairing, [L3], [L4]. The duality theorem of [M] implies the isomorphism of right Λ-modules:

$$\bar{H}_q(\tilde{X}) \approx H_e^{n+2-q}(\tilde{X}, \partial\tilde{X}) \tag{1}$$

where $\bar{H}_q(\tilde{X})$ denotes the right Λ-module defined from the original left Λ-structure by the usual means. $H_e^*(\tilde{X}, \partial\tilde{X})$ is the homology of the cochain complex $\mathrm{Hom}_\Lambda(C_*(\tilde{X}, \partial\tilde{X}), \Lambda)$. Then there is an exact sequence for $0 \leq q \leq n$:

$$0 \to \mathrm{Ext}^2_\Lambda(A_{n-q}, \Lambda) \to \bar{A}_q \to \mathrm{Ext}^1_\Lambda(A_{n+1-q}, \Lambda) \to 0. \tag{2}$$

This follows from the isomorphism (1), the trivial nature of $\partial\tilde{X}$, the universal coefficient spectral sequence, the homological dimension of Λ, and the fact that A_q is a Λ-torsion module. For an Alexander module A_i, $\mathrm{Ext}^2_\Lambda(A_i, \Lambda)$ is a $\mathbf{Z}$-torsion module and depends only on T_i, while $\mathrm{Ext}^1_\Lambda(A_i, \Lambda)$ is $\mathbf{Z}$-torsion free and depends only on A_i/T_i; therefore

$$\bar{T}_q \approx \mathrm{Ext}^2_\Lambda(T_{n-q}, \Lambda) \quad \text{for} \quad 0 \leq q \leq n. \tag{3}$$

Now for any finite Λ-module T, there is a canonical isomorphism of Λ-modules:

$$\mathrm{Ext}^2_\Lambda(T, \Lambda) \approx \mathrm{Hom}_{\mathbf{Z}}(T, \mathbf{Q}/\mathbf{Z}). \tag{4}$$

In fact T_{n-q} is finite, so that one derives the Levine pairing by combining the isomorphism (3), (4):

$$[\,,\,]: T_q \times T_{n-q} \text{—} \mathbf{Q}/\mathbf{Z}.$$

In case $n = 2q$, it satisfies the following four properties:

a) $\mathbf{Z}$-linear: $[m\alpha, \beta] = [\alpha, m\beta] = m[\alpha, \beta]$ for $m \in \mathbf{Z}$ $\alpha, \beta \in T_q$
b) conjugate self-adjoint: $[\lambda\alpha, \beta] = [\alpha, \bar{\lambda}\beta]$ for $\lambda \in \Lambda$

c) non-singular: the adjoint to $[,]$ is bijective as a homomorphism: $T_q \to \mathrm{Hom}_{\mathbf{Z}}(T_q, \mathbf{Q}/\mathbf{Z})$

d) $(-1)^{q+1}$-symmetric: $[\alpha, \beta] = (-1)^{q+1}[\beta, \alpha]$.

Remark 3. This is an almost complete algebraic characterization of the Levine pairing. In fact, for any pairing $[,]$ on T_q satisfying a), b), c) and d), there exists a $2q$-knot with the Levine pairing $[,]$ provided $q \geq 2$, [$L3$], [$L4$].

We shall now give an alternative description of Levine pairings by making use of the τ-Seifert form defined by M. A. Gutiérrez.

A *Seifert manifold* V of a $2q$-knot K is a smooth oriented submanifold of S^{2q+2} which is bounded by K. Writing τ for the torsion subgroup, we define the τ-intersection pairing (classically called the linking number) $I: \tau H_q(V) \otimes \tau H_q(V) \to \mathbf{Q}/\mathbf{Z}$ and the τ-linking pairing $L: \tau H_q(V) \otimes \tau H_q(S^{2q+2} - V) \to \mathbf{Q}/\mathbf{Z}$ as follows. Let $\alpha \in \tau H_q(V)$ have order d. Represent α by a q-chain ξ, and let ζ be a $(q+1)$-chain such that $\partial \zeta = d \cdot \xi$. Then if $\beta \in \tau H_q(V)$ is represented by a q-chain η, we have $I(\alpha, \beta) = \mathrm{Int}(\zeta, \eta)/d \pmod 1$, where $\mathrm{Int}(,)$ is the usual intersection number in V. And if $\gamma \in \tau H_q(S^{2q+2} - V)$ is represented by a q-chain μ with $\partial \rho = \mu$ for some $(q+1)$-chain ρ in S^{2q+2}, we have $L(\alpha, \gamma) = \mathrm{Int}(\zeta, \rho)/d \pmod 1$, where $\mathrm{Int}(,)$ is now the intersection number in S^{2q+2}. Then the *τ-Seifert form* of V

$$\theta: \tau H_q(V) \otimes \tau H_q(V) \to \mathbf{Q}/\mathbf{Z}$$

is defined by letting $\theta(\alpha \otimes \beta)$ be the τ-linking number $L(\alpha, \beta_+)$ where β_+ is the translate in the positive normal direction off V of the cycle β.

A finite abelian group G splits as the direct sum of its Sylow subgroups. Since these are clearly orthogonal with respect to any form on G, the whole problem splits also geometrically. This refers to arguments given later in the paper. Now, we assume that $\tau H_q(V)$ is a p-group throughout this paper. Let $\{\alpha_i\}_{i=1}^k$ be a basis of $\tau H_q(V)$ having order p_i with elementary divisor $p_1 \mid p_2 \mid \cdots \mid p_k$ and $\{\beta_i\}_{i=1}^k \in \tau H_q(S^{2q+2} - V)$ be the Alexander dual basis of $\{\alpha_i\}$. The embeddings $\kappa, \iota: V \to S^{2q+2} - V$ are defined by $\pm\varepsilon$-pushes. Then, for the maps $\kappa_*, \iota_*: \tau H_q(V) \to \tau H_q(S^{2q+2} - V)$ induced by κ, ι, we shall give the matrix presentations:

$$\kappa_* \alpha_j = \sum_{i=1}^k m_{ij}^+ \beta_i \qquad j = 1, 2, \ldots, k$$

$$\iota_* \alpha_j = \sum_{i=1}^k m_{ij}^- \beta_i \qquad j = 1, 2, \ldots, k.$$

Indeed, the matrices $M_\pm = (m_{ij}^\pm)$ are not uniquely determined as integral entries, but are uniquely determined modulo Δ, where $\Delta = P \cdot J$,

$$P = \begin{pmatrix} p_1 & & & & \bigcirc \\ & p_2 & & & \\ & & \cdot & & \\ & & & \cdot & \\ \bigcirc & & & & p_k \end{pmatrix} \quad \text{and} \quad J = \begin{pmatrix} 1, & 1, & \dots, & 1 \\ 1, & \cdot & & \cdot \\ & & & \cdot \\ & & & \cdot \\ & & & \cdot \\ 1, & & & 1 \end{pmatrix}$$

And if $M_\theta = (p_i \cdot \theta(\alpha_i \otimes \alpha_j))$, then $M_+ = M_\theta \pmod{\Delta}$ and $M_- = M'_\theta \pmod{\Delta}$ where $M'_\theta = P \cdot {}^t M_\theta \cdot P^{-1}$. The result of [G] is

PROPOSITION 1 (M. A. Gutiérrez). *A presentation matrix of T_q is given by $(t \cdot M_\theta + (-1)^{q+1} M'_\theta, P)$. More precisely, there is an exact sequence*

$$\bigoplus^{2k} \Lambda \xrightarrow{d} \bigoplus^{k} \Lambda \longrightarrow T_q \longrightarrow 0$$

such that a matrix presentation of d with respect to suitable bases is given by the above.

In general, the τ-Seifert form of V may be singular. But in our case, if V is minimal, its Seifert form will be non-singular. Indeed, recall that a Seifert manifold is minimal if $\kappa_*, \iota_* : H_i(V) \to H_i(S^{2q+2} - V)$ are injective for all i. The following is then implicit in [L1];

PROPOSITION 2 (J. Levine). *Let K be a simple n-knot and $n \geq 4$. Then there exists a minimal Seifert manifold for K. Moreover, we can choose it as being simply connected.*

Let now K be a finite simple $2q$-knot and let V be a minimal Seifert manifold for K. Then $H_q(V)$ is finite, otherwise $H_q(\tilde{V})$ would not be. As $H_q(S^{2q+2} - V)$ is abstractly isomorphic to $H_q(V)$ and as κ_* and ι_* are injective, they are necessarily isomorphisms. This implies immediately that the τ-Seifert form is non-singular.

Therefore there is an inverse isomorphism κ_*^{-1}. $H_q(S^{2q+2} - V) \to H_q(V)$. Let M_θ^{-1} be the representation matrix of κ_*^{-1} with respect to bases $\{\beta_i\}$ and $\{\alpha_i\}$, then $M_\theta \cdot M_\theta^{-1} = M_\theta^{-1} \cdot M_\theta = E \pmod{\Delta}$ where E is the identity matrix. The square matrix $I_V = (I(\alpha_i, \alpha_j))$ with entries in $\mathbf{Q}/\mathbf{Z}$ is the τ-intersection matrix. By Proposition 1, an element of T_q can be represented by a column vector of $\bigoplus^k \Lambda$. The *quotient form* $[\,,]_\theta$ on T_q determined by θ with values in $\mathbf{Q}/\mathbf{Z}$ is defined by letting

$$[\alpha, \beta]_\theta = \text{the constant term of } {}^t\bar{u} \cdot \tilde{U}_\theta \cdot v \pmod 1$$

where u and v are representatives of α and β. Here $\tilde{U}_\theta$ is the square matrix with entries in $\mathbf{Q}/\mathbf{Z}$ $[[t, t^{-1}]]$ defined as follows: let U_θ be the $\mathbf{Q}/\mathbf{Z}$-matrix $U_\theta = {}^tM_\theta^* \cdot I_V \cdot M_\theta^*$, where $M_\theta^* = (M_\theta')^{-1}$ and A_θ be the $\mathbf{Z}$-matrix $A_\theta = (-1)^q M_\theta' \cdot M_\theta^{-1}$ (mod Δ). Then $\tilde{U}_\theta = \sum_{n=-\infty}^{\infty} t^n \cdot ({}^tA_\theta)^n \cdot U_\theta$.

We recall the formula $[G]$:

$$\theta(\alpha \otimes \beta) + (-1)^{q+1}\theta(\beta \otimes \alpha) = -I(\alpha, \beta) \pmod 1.$$

So that $M_\theta + (-1)^{q+1}M_\theta' = -P \cdot I_V$ (mod Δ). This implies that the quotient form $[\,,]_\theta$ is well defined and satisfies the four properties a), b), c) and d). And because of Levine's observation, it coincides with the Levine pairing $[\,,]$ of K (see[$L3$], Proposition 7.1).

Remark 4. The above computation method of the Levine pairing is essentially due to the minimality of a Seifert manifold. And the simplicity of the knot, the existence of a fibration and the finiteness of $\pi_q(X)$ are inessential.

§3. Proof of Theorem 1

From now on, we shall prove Theorem 1 in three steps. The first is devoted to show that an isometry of Levine pairings induces an isometry of τ-Seifert forms. And in the next step, we construct an isotopy between Seifert manifolds when q is odd. The case q even is studied in the last.

3.1. Now, let K_0 and K_1 be odd finite simple $2q$-knots with isometric Levine pairings. V and W are minimal, simply connected Seifert manifolds of K_0 and K_1, and θ, η are their τ-Seifert forms.

LEMMA 1. *If the quotient form $[\,,]_\theta$ is isometric to $[\,,]_\eta$, then θ is isometric to η.*

Proof. Let T_q and T_q' be q-th Alexander modules of K_0 and K_1. Then by assumption, there is a Λ-isomorphism $f: T_q \to T_q'$ such that $[\alpha, \beta]_\theta = [f(\alpha), f(\beta)]_\eta$ for each $\alpha, \beta \in T_q$. For some bases $\{\alpha_i\}$, $\{\alpha_i'\}$ of T_q, T_q', f is of the form $f(\alpha_i) = \Sigma a_{ij}\alpha_j'$. Since it can be regarded as a group isomorphism: $H_q(V) \to H_q(W)$, $\{f(\alpha_i)\}$ is a basis of $H_q(W)$. If M_η is a representative matrix of η with respect to the basis $\{f(\alpha_i)\}$, then $U_\theta = U_\eta$ and $A_\theta = A_\eta$ because f is a Λ-isomorphism and an isometry. We recall that $M_\theta + (-1)^{q+1}M_\theta' = -P \cdot I_V$ (mod Δ), so that

$$\begin{aligned} A_\theta &= E + P \cdot I_V \cdot M_\theta^{-1} \pmod{\Delta} \\ &= E + (-1)^q M_\theta \cdot P \cdot U_\theta \cdot A_\theta \pmod{\Delta}. \end{aligned}$$

Similarly $A_\eta = E + (-1)^q M_\eta \cdot P \cdot U_\eta \cdot A_\eta \pmod{\Delta}$, thus we obtain the identity $M_\theta \cdot P \cdot U_\theta \cdot A_\theta = M_\eta \cdot P \cdot U_\eta \cdot A_\eta \pmod{\Delta}$. Now, A_θ is invertible in the sense of modulo Δ. And so is $P \cdot U_\theta$ because the τ-intersection form of V is non-singular. Therefore we have that $M_\theta = M_\eta$. This implies that $f: H_q(V) \to H_q(W)$ is an isometry, completing the proof.

Remark 5. In the argument above, simplicity, oddness and finiteness of knots are not essential, but minimality of V is.

In the following, we shall constract the isotopy between Seifert manifolds V and W, by virtue of the dimensional condition $q \geq 4$. Indeed, if $q \geq 4$ almost all our arguments are in the metastable range and in that range homotopy implies isotopy by Haefliger's theorem [H].

3.2. Now we assume that q is odd. Then the τ-intersection form of V is symmetric and hence diagonal because $H_q(V)$ has no 2-torsion. Namely there is a basis $\{\alpha_i\}$ of $H_q(V)$ such that $I(\alpha_i, \alpha_j) = \delta_{ij} c_i / p_i$, where c_i is an integer co-prime to p_i, and δ_{ij} is Kronecker's delta. A decomposition of V can be obtained as follows. Let $A_j = S^q \times D^{q+1} \cup_{f_i} h$ where h is a $(q+1)$-handle and f_i is the attaching map of h, i.e., $f_j: \partial D^{q+1} \times D^q \to \partial(S^q \times D^{q+1})$. The homotopy group $\pi_q(S^q \times \partial D^{q+1})$ is a free abelian group generated by two elements $x = S^q \times \{pt\}$ and $y = \{pt\} \times \partial D^{q+1}$. And the homotopy class of $f_j \mid \partial D^{q+1} \times \{0\}$ is $p_j x + q_j y$, where $q_j = c_j \pmod{p_j}$. The manifold A_j is fundamental for V. In fact, it follows from the high connectivity and the parallelizability of V that it is diffeomorphic to the boundary connected sum $\natural_{j=1}^k A_j$ (see [W]). The Seifert manifold W also admits a similar decomposition. More precisely, there is a basis $\{\alpha_i'\}$ of $H_q(W)$ such that $\theta(\alpha_i \otimes \alpha_j) = \eta(\alpha_i' \otimes \alpha_j')$ because θ is isometric to η. Then W has a decomposition $\natural_{j=1}^k A_j'$ associated to the basis $\{\alpha_i'\}$.

We next show how to isotopically deform A_j onto A_j' in S^{2q+2}. For this reason, a number of preliminaries are necessary.

A *q-annulus* will be a smooth oriented submanifold R of S^{2q+2}, where R is diffeomorphic to $S^q \times D^{q+1}$. A q-annulus has a trivial disc bundle structure on $S^q = S^q \times \{0\}$. Let ν be the positive unit normal field to R on S^q. Then by the tubular neighborhood theorem, R can be considered as the orthogonal complement of ν in a normal disc bundle neighborhood of S^q in S^{2q+2}. Let $R_i = S_i^q \times D_i^{q+1}$, $i = 0, 1$, be q-annuli. R_0 and R_1 are isotopic if there is an ambient isotopy χ_t of R_0 to R_1 such that $\chi_0 = \mathrm{id}$, $\chi_1(S_0^q) = S_1^q$ and χ_1 is a bundle map. However, by the unknotting theorem, the tubular neighborhood theorem and the fact $\pi_q(S^{q+1}) = 0$, all the q-annuli are isotopic if $q \geq 1$. The condition q odd is not necessary for this. Next, we consider a q-annulus with a non-zero section (R, s), where s is a section of the associated sphere bundle ∂R. We shall say that (R_0, s_0)

is isotopic to (R_1, s_1) if there is an isotopy χ_t of R_0 to R_1 such that $\chi_1 \circ s_0 = s_1 \circ \chi_1 | S_0^q$. Isotopy $\sim$ is an equivalence relation on Ω, the set of q-annuli with a non-zero section. Clearly it is a necessary condition for (R_i, s_i), $i=0, 1$, to be isotopic that the characteristic class of the normal bundle of s_0 in ∂R_0 coincides with that of s_1. If q is odd, $\neq 1, 3, 7$, the image of ∂: $\pi_q(S^q) \to \pi_{q-1}(SO_q)$ is $\mathbf{Z}_2$. So that $\Omega/\sim$ has at least two classes in this case. In fact:

LEMMA 2. *There is a bijective map* $\phi : \Omega/\sim \to \{0, 1\}$ *if* q *is odd and* $q \geq 1$.

Proof. First of all, ϕ will be defined as follows. We set $S^{2q+2} = \{(x_0, x_1, \ldots, x_{2q+2}) \in R^{2q+3} \mid x_0^2 + x_1^2 + \cdots + x_{2q+2}^2 = 1\}$, the equator $S^{2q+1} = \{x \in S^{2q+2} \mid x_{2q+2} = 0\}$ and $S^q = \{x \in S^{2q+2} \mid x_{q+1} = x_{q+2} = \cdots = x_{2q+2} = 0\}$. R is the normal unit disc bundle of S^q in S^{2q+1}. For any q-annulus R', there is an isotopy χ_t of R' to R. Then $\phi((R', s))$ is the modulo two reduction of the linking number between S^q and $\chi_1 \circ s \circ \chi_1^{-1}(S^q)$ in S^{2q+1}.

We must prove two points: that ϕ is well-defined and that it is bijective. Suppose that there is an isotopy f_t of (R_0, s_0) to (R_1, s_1). Without loss of generality, we may assume that $(R_0, s_0) = (R, s)$ where s is a suitable section of R. Let g_t be the isotopy of R_1 to R with which we define the number $\phi((R_1, s_1))$. Combining these,

$$\chi_t = \begin{cases} f_{2t} & \text{for } \ 0 \leq t \leq 1/2 \\ g_{2t-1} & \text{for } \ 1/2 \leq t \leq 1 \end{cases}$$

is an isotopy of the q-annulus R to itself. Now, the following diagram will be studied.

$$\begin{tikzcd} \pi_q(SO_{q+1}) \arrow[r, "i*"] \arrow[dr, "p*"'] & \pi_q(SO_{q+2}) \arrow[r] & 0 \\ & \pi_q(S^q) \approx \mathbf{Z} & \end{tikzcd}$$

The framing of R differs from that of $\chi_1(R)$ by an element of $\pi_q(SO_{q+1})$. But it is null homotopic in SO_{q+2}. In other words, it is an element of Ker i_*. If q is odd, a generator of Ker i_* is mapped to 2 by p_*, so that the difference between the linking number of $s(S^q)$ and $g_1 \circ s_1 \circ g_1^{-1}(S^q)$ with S^q in S^{2q+1} is even. Thus ϕ is well-defined.

Conversely, we shall suppose that $\phi((R_0, s_0)) = \phi((R_1, s_1))$. Then by using a generator of Ker i_*, the isotopy between them can be easily constructed. Therefore ϕ is bijective. This completes the proof.

Now, we shall define two isotopies which play a key role to deform a $(q+1)$-handle h to h'. Let a q-annulus R be as in Lemma 2 and N be the normal disc bundle of S^q in S^{2q+2}. Then we take a trivialization $T: N \to S^q \times D^{q+2}$ such that $p \circ T(R) = D^{q+1}$ where p denotes the projection onto the second factor. Here $D^{q+2} = \{(y_1, y_2, \ldots, y_{q+2}) \in R^{q+2} \mid y_1^2 + y_2^2 + \cdots + y_{q+2}^2 \leq 1\}$ and $D^{q+1} = \{y \in D^{q+2} \mid y_{q+2} = 0\}$. As was seen before, a q-annulus R is the orthogonal complement of a positive unit normal field ν to R on S^q in N. When we regard ν as a map of S^q to ∂D^{q+2} by composing T and p, it is a constant map to the base point $* = (0, 0, \ldots, 0, 1)$ of ∂D^{q+2}. A homotopical deformation of ν as a section induces an ambient isotopy of R. Let $F_i : S^q \times I \to \partial D^{q+2}$, $i = 0, 1$, be homotopical deformations of ν which satisfy two properties.

(1) $F_i(x, 0) = F_i(x, 1) = F_i(b, t) = *$ for each $x \in S^q$, $t \in [0, 1]$ and $i = 0, 1$, where b is the base point of S^q.

This property implies that F_i can be considered as a map of S^{q+1} to ∂D^{q+2}.

(2) F_0 is a degree one map of S^{q+1}. Im F_1 is included in $S^q = \{y \in D^{q+2} \mid y_1 = 0\}$ and F_1 represents the non-trivial element of $\pi_{q+1}(S^q) \approx \mathbf{Z}_2$, $(q \geq 3)$.

Let J_i be the isotopy of R induced by F_i. In case q is odd, J_0 is just the one which is induced by a generator of Ker i_*.

Coming back to our previous program, we shall isotopically deform A_1 onto A_1', provided $q \geq 5$. Using q-annuli, $A_1 = R_1 \cup_{f_1} h_1$ and $A_1' = R_1' \cup_{f_1'} h_1'$. There always exists an isotopy from R_1 to R_1'. Then the attaching sphere of h_1 represents a homotopy class $p_1 x + q_1 y \in \pi_q(\partial R_1)$. By hypothesis, the attaching sphere of h_1' represents $p_1' x + q_1' y$, where $p_1 = p_1'$ and $q_1 \equiv q_1' \pmod{p_1}$. Since the normal bundle of $f_1(\partial D^{q+1} \times \{0\})$ is trivial, $\partial(\operatorname{Int}(p_1 x, q_1 y)) = 0$. Here, the intersection number is regarded as an element of $\pi_q(S^q)$ and ∂ is the boundary map: $\pi_q(S^q) \to \pi_{q-1}(SO_q)$. If q is odd and $\neq 7$, then the image of ∂ is $\mathbf{Z}_2$, so that q_1 is even because p_1 is odd. Therefore $q_1 \equiv q_1' \pmod{2p_1}$. If $q = 7$, this is true by a general argument: namely Lemma 2 shows that the modulo two reduction of q_1 is the primary obstruction to thickening the core of h_1 in S^{2q+2}. Now, $q_1 \equiv q_1' \pmod{2p_1}$ implies that there is an integer n such that $q_1 + 2np_1 = q_1'$. By Lemma 2, there is an isotopy $(J_0)^n$ from (R_1, x) to $(R_1, x + 2ny)$. Thus it changes the homotopy class of $f_1(\partial D^{q+1} \times \{0\})$ to $p_1(x + 2ny) + q_1 y = p_1' x + q_1' y$. After all, we can isotopically deform R_1 onto R_1' so that the attaching sphere of h_1 coincides with that of h_1'.

Next, we set the manifold M as being S^{2q+2}-interior N. Then there is a homotopy exact sequence:

$$0 \to \pi_{q+1}(M) \to \pi_{q+1}(M, \tilde{S}^q) \to \pi_q(\tilde{S}^q) \to 0$$

where $\tilde{S}^q$ is the attaching sphere of h_1. M has the same homotopy type as S^{q+1}, so that $\pi_{q+1}(M, \tilde{S}^q)$ is free abelian of rank two. And the core $C_1(C_1')$ of $h_1(h_1')$ represents an element $c(c')$ of $\pi_{q+1}(M, \tilde{S}^q)$. Rounding the edge, $C_1 \cup C_1'$ is considered as an immersed $(q+1)$-sphere with some orientation in interior M. Then the self-intersection number of it is identified with $p_1 \cdot (c-c') \in \pi_{q+1}(M) \approx \mathbf{Z}$ as an integer. On the other hand, the normal bundle of $C_1 \cup C_1'$ is trivial because C_1 and C_1' are the core of $(q+1)$-handles. Therefore the self-intersection number of it must be zero. Thus c is in fact the same element as c'. This shows that C_1 is isotopically deformed onto C_1'.

Let $\mu(\mu')$ be a positive unit normal field to $h_1(h_1')$ on $C_1(C_1')$. The $(q+1)$-handle h_1 is the orthogonal complement of μ in a normal disc bundle of C_1 in S^{2q+2}. Since $\mu = \mu'$ along ∂C_1, μ differs from μ' by an element of $\pi_{q+1}(S^q) \approx \mathbf{Z}_2$. If $\mu - \mu' = 0$, there is a homotopy of μ to μ' relative to ∂C_1, so that h_1 is isotopically deformed to h_1'. If not so, we deform R_1 with an isotopy J_1. It does not change the homotopy class of ∂C_1, but changes the homotopy class of μ into the other element if p_1 is odd. And then, for the new field μ, $\mu - \mu' = 0$. Thus we obtain the required isotopy from A_1 to A_1'.

Remark 6. While constructing the above isotopy, we only used the information that $I_V(\alpha_1, \alpha_1) = I_W(\alpha_1', \alpha_1')$. But in fact, the identity $2 \cdot \theta(\alpha_1 \otimes \alpha_1) = I_V(\alpha_1, \alpha_1)$ follows from Gutiérrez's formula, and this implies that the diagonal entry of the τ-Seifert form is completely determined by its self τ-intersection number, if p_1 is odd.

Next we shall construct the deformation of A_2 onto A_2' in $S^{2q+2} - A_1$. The space $S^{2q+2} - A_1$ has the same homotopy type as A_1, so that $\pi_q(S^{2q+2} - A_1) \approx \mathbf{Z}_{p_1}$. The core of the q-annulus $R_2(R_2')$ of $A_2(A_2')$ represents an element $r(r')$ of $\pi_q(S^{2q+2} - A_1)$ and $r - r'$ can be identified with $\theta(\alpha_1 \otimes \alpha_2) - \eta(\alpha_1' \otimes \alpha_2') = 0$. Therefore there is an isotopy from R_2 to R_2' by which the attaching sphere of h_2 coincides with that of h_2'. Put $M = S^{2q+2}$-interior $A_1 \vee N$, where N is the normal disc bundle of R_2 and $\vee$ denotes the one point union. Then there is an exact sequence:

$$0 \to \pi_{q+1}(M) \to \pi_{q+1}(M, \tilde{S}^q) \to \pi_q(\tilde{S}^q) \to 0$$

where $\tilde{S}^q$ is the attaching sphere of h_2. I have $\pi_{q+1}(A_1) = 0$, because $A_1 \simeq S^q \cup_{p_1} D^{q+1}$ and p_1 is odd (the left distributivity law applies). So $\pi_{q+1}(M) \approx \mathbf{Z}$ and $\pi_{q+1}(M, \tilde{S}^q) \approx \mathbf{Z} \oplus \mathbf{Z}$. Using the same argument as before we can isotopically deform h_2 onto h_2' and so that A_2 onto A_2' in $S^{2q+2} - A_1$.

Proceeding inductively, we finally obtain the required isotopy of V to W. This completes the proof of Theorem 1 in case q is odd.

3.3 Here, we shall give a proof of Theorem 1 when q is even. If so, the

τ-intersection form of V is skew-symmetric and $H_q(V)$ has a basis $\{\alpha_i\}_{i=1}^{2k}$ such that the order $p_{2i-1}=p_{2i}$ and for $i<j$,

$$I(\alpha_i,\alpha_j)=\begin{cases}1/p_i & \text{if } i=2n-1 \text{ and } j=2n\\ 0 & \text{otherwise.}\end{cases}$$

A decomposition of V can be obtained as follows. Let a fundamental manifold B_j be $S^q\times D^{q+1}\natural S^q\times D^{q+1}\cup_{f_{2j-1}}h\cup_{f_{2j}}h$, where h is a $(q+1)$-handle and f_{2j-1} (f_{2j}) is an attaching map. Now, $\pi_q(\partial(S^q\times D^{q+1}\natural S^q\times D^{q+1}))$ is generated by the natural four elements x_1, y_1, x_2 and y_2 of the first and the second factors. Choose the attaching sphere of h_{2j-1} (resp. h_{2j}) to represent the element $p_{2j-1}x_1+y_2$ (resp. $p_{2j}x_2-y_1$) of $\pi_q(\partial(S^q\times D^{q+1}\natural S^q\times D^{q+1}))$. Then V is diffeomorphic to the boundary connected sum $\natural_{j=1}^{k}B_j$ (see [W]).

To prove Theorem 1, we shall consider the fundamental manifold B_j from an alternative viewpoint. As a submanifold of S^{2q+2}, $S^q\times D^{q+1}\natural S^q\times D^{q+1}$ is just a q-annulus R_j with a q-handle H_j. Moreover, we may assume that the image of f_{2j-1} is on ∂R_j and the attaching sphere of H_j is on the complement $\partial R_j-\operatorname{Im} f_{2j-1}$. After all, $B_j=R_j\cup H_j\cup_{f_{2j-1}}h_{2j-1}\cup_{f_{2j}}h_{2j}$ as a submanifold of S^{2q+2}. The manifold W also admits a similar decomposition $\natural_{j=1}^{k}B'_j$ associated to an isometric basis $\{\alpha'_i\}_{i=1}^{2k}$.

We next show how to deform B_1 onto B'_1. As was seen in 3.2, there always exists an isotopy from R_1 to R'_1. The attaching sphere of h_1 (resp. h'_1) represents a homotopy class p_1x+q_1y (resp. $p'_1x+q'_1y$) of $\pi_q(\partial R_1)$, where $x=S^q\times\{pt\}$ and $y=\{pt\}\times\partial D^{q+1}$. If q is even, the boundary homomorphism $\partial:\pi_q(S^q)\to\pi_{q-1}(SO_q)$ is injective. Since the normal bundle of the attaching sphere of $h_1(h'_1)$ is trivial, $\partial(\operatorname{Int}(p_1x,q_1y))=0$ (resp. $\partial(\operatorname{Int}(p'_1x,q'_1y))=0$). This implies $q_1=q'_1=0$. Therefore, under this isotopy the image of f_1 coincides with that of f'_1.

Let N be a normal disc bundle of R_1 in S^{2q+2}, and put $M=S^{2q+2}$-interior N. Then there is a split exact sequence:

$$0\to\pi_{q+1}(M)\to\pi_{q+1}(M,\tilde{S}^q)\to\pi_q(\tilde{S}^q)\to 0$$

where $\tilde{S}^q$ is the attaching sphere of $h_1(h'_1)$. Let u, v be a basis of $\pi_{q+1}(M,\tilde{S}^q)$, corresponding to the above splitting. The core $C_1(C'_1)$ of $h_1(h'_1)$ represents an element $u+q_1v(u+q'_1v)$ of $\pi_{q+1}(M,\tilde{S}^q)$, where $q_1\equiv p_1\cdot\theta(\alpha_1\otimes\alpha_1)\pmod{p_1}$ $(q'_1\equiv p'_1\cdot\eta(\alpha'_1\otimes\alpha'_1)\pmod{p'_1})$. This shows that there is an integer n such that $q'_1=q_1+np_1$. Then we deform R_1 by $(J_0)^n$, so that it changes the homotopy class of C_1 into $u+(q_1+np_1)v=u+q'_1v$. Thus we obtain an isotopy of $R_1\cup C_1$ to $R'_1\cup C'_1$. The rest of the procedure to deform $R_1\cup h_1$ onto $R'_1\cup h'_1$ is the same as in 3.2.

The attaching sphere of H_1 is on $\partial R_1-\operatorname{Im} f_1$ and represents an element of

$\pi_{q-1}(\partial R_1 - \operatorname{Im} f_1) \approx \mathbf{Z}_{p_1}$. It differs from that of H_1' by $I_V(\alpha_1, \alpha_2) - I_W(\alpha_1', \alpha_2') = 0$. So that we may assume that these coincide. Using the previous decomposition of $B_1(B_1')$, the core $C_1(C_1')$ of $H_1(H_1')$ is naturally extended to a q-sphere $S(S')$. Let $S_+(S_+')$ be the translate in the positive normal direction off $V(W)$ of $S(S')$. It represents an element of $\pi_q(S^{2q+2} - R_1 \cup h_1) \approx \mathbf{Z}_{p_1}$, and the difference between those is identified with $\theta(\alpha_1 \otimes \alpha_2) - \eta(\alpha_1' \otimes \alpha_2') = 0$. Therefore S_+ is isotopic to S_+' in $S^{2q+2} - R_1 \cup h_1$, and moreover C_1 is isotopic to C_1' keeping ∂C_1 fixed. Afterward, using the same procedure as the one we have used in deforming A_2 onto A_2', we obtain an isotopy of B_1 to B_1'.

The rest of the proof is done inductively as in the case q odd. Thus we finally obtain the required isotopy of V to W. This completes the proof of Theorem 1.

§4. Addendum

Here, we shall expose without proof some geometrical properties of some fibered knots. By [$B\&L$], a necessary and sufficient condition for a simple $2q$-knot K to be fibered is that $\pi_q(X)$ is finitely generated, where X is the complement of K. Throughout this section, a knot K is always fibered. Definitions can be found in our previous article [Ko]. Then our result is

THEOREM 2. *Let K be an odd simple fibered $2q$-knot, and $q \geq 4$. Then the isotopy class of K is completely determined by the first, the second and the τ-Seifert forms.*

The crux of the proof is how to choose a basis of $H_q(V)$, where V is a minimal Seifert manifold of K. Namely, setting $\{\alpha_i\}$ as being a basis of $\tau H_q(V)$, we can choose a basis $\{\beta_j\}$ of the torsion free part of $H_q(V)$ such that they are null-homologous in the complement of the chains $\{\zeta_i\}$ in S^{2q+2} where $\partial \zeta_i = p_i \cdot \alpha_i$. The reason why we can do this is the non-singularity of the τ-Seifert form of V. Choosing a nice decomposition of V with respect to the basis $\{\alpha_i, \beta_j\}$ of $H_q(V)$, we shall use the same procedure as in the proof of Theorem 1 and [Ko]. This supplies the proof of Theorem 2. Now, the following corollaries are immediately obtainable from Theorem 2.

COROLLARY 1. *If a $2q$-knot K is as above, then there is a unique splitting $K = K_F \# K_T$ such that $\pi_q(X_F)$ is torsion free and $\pi_q(X_T)$ is finite, where $\#$ denotes the knot connected sum and $X_F(X_T)$ is the complement of a simple knot $K_F(K_T)$.*

COROLLARY 2. *Let K be as above and $p, p' : X \to S^1$ be fibrations of it. Then there exists a diffeomorphism f of S^{2q+2} such that $p' = p \circ f \mid X$.*

REFERENCES

[B & L] BROWDER, W., LEVINE, J., *Fibering manifolds over a circle*. Comment. Math. Helv. *40*, (1965) 153–160.

[G] GUTIÉRREZ, M., *On knot modules*. Inventiones math. *17*, (1972) 329–335.

[H] HAEFLIGER, A., Plongements différentiables de variétés dans variétés. Comment. Math. Helv. *36*, (1961) 47–82.

[K1] KEARTON, C., *Classification of simple knots by Blanchfield duality*. Bull. of the A.M.S. *79*, (1973) 952–955.

[K2] KEARTON, C., *A classification of some even dimensional knots*. Topology *15*, (1976) 363–373.

[Ko] KOJIMA, S., *A classification of some even dimensional fibered knots*. J. Fac. Sci. Univ. Tokyo *24*, (1977) 671–683.

[L1] LEVINE, J. *Unknotting spheres in codimension two*. Topology *4*, (1965) 9–16.

[L2] LEVINE, J., *An algebraic classification of some knots of codimension two*. Comment. Math. Helv. *45*, (1970) 185–198.

[L3] LEVINE, J., *Knot modules, I*. Trans. of the A.M.S. *229*, (1977) 1–50.

[L4] LEVINE, J., *Knot modules*. Annales of Math. Studies 84 Princeton Univ. Press (1975) 25–34.

[M] MILNOR, J., *A duality theorem for Reidemeister torsion*. Ann. of Math. *76*, (1962) 137–147.

[T] TROTTER, H., *On s-equivalence of Seifert matrices*. Inventiones math. *20*, (1973) 173–207.

[W] WALL, C. T. C., *Classification problems in differential topology–VI*. Topology *6*, (1967) 273–296.

Department of Mathematics
Faculty of Science
University of Tokyo

current address
Department of Mathematics
Tokyo Metropolitan University

Received October 22, 1977/November 2, 1978.