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## A homological characterization of foliations consisting of minimal surfaces

by DENNIS SULLIVAN

Say that  $p$ -dimensional foliation of a compact  $n$ -manifold is *geometrically taut* if there is a Riemann metric for which the leaves become minimal surfaces. Say that an oriented  $p$ -dimensional foliation is *homologically taut* if no foliation cycle  $[S_1]$  constructed from an invariant transverse measure  $[P]$  and  $[RS]$  is approximately the boundary of a  $(p+1)$ -chain tangent to the foliation.

**THEOREM.** For an orientable foliation to be geometrically taut it is necessary and sufficient that it be homologically taut.

*Proof.* The ingredients are Rummler's calculation  $[R]$ , an algebraic operation "purification" preserving a differential condition, Stokes theorem, and the Hahn Banach theorem in the general set-up of  $[S_1]$ .

*Rummler's calculation.* A  $p$ -dimensional foliation  $\mathcal{F}$  of a piece of Riemannian manifold has leaves which are minimal surfaces if and only if the characteristic  $p$ -form is relatively  $\mathcal{F}$ -closed. The characteristic  $p$ -form is obtained from the oriented volume form of  $\mathcal{F}$  by orthogonal projection of  $p$ -vectors onto the tangent planes of  $\mathcal{F}$ . A  $p$ -form is relatively  $\mathcal{F}$ -closed if its restriction to every  $(p+1)$ -manifold tangent to  $\mathcal{F}$  is closed.

*Purification.* To each  $p$ -form  $\omega$  on a vector space positive on an oriented  $p$ -dimensional subspace  $F$  associates the pair

$$(P_\omega, \omega/F) = (\text{projection onto } F, \text{volume form on } F).$$

Here  $/$  means restriction and  $P_\omega$  is defined by the equation

$$P_\omega(v) \wedge (\omega/F) = (v \wedge \omega)/F$$

where  $\wedge$  means contraction. The pure form  $\tilde{\omega} = P_\omega^*(\omega/F)$  is called the purification of  $\omega$ .

*Hahn-Banach.* An orientation of a foliation  $\mathcal{F}$  allows one to regard a transversal invariant measure [RS] and  $[S_1]$  as a  $p$ -current and these form precisely the intersection of the closed  $p$ -currents with the “compact cone” of foliation currents (convex combinations of oriented tangent  $p$ -vectors) Theorem I.13  $[S_1]$ . Homological tautness means the closed subspace  $S$  generated by boundaries of  $(p+1)$ -chains tangent to  $\mathcal{F}$  strictly supports the intersection cone of foliation cycles. Hahn–Banach then applied as in Theorem I.7  $[S_1]$  allows us to construct a  $p$ -form  $\omega$  positive on  $\mathcal{F}$  and which annihilates this space  $S$  of boundaries. Such a form is relatively  $\mathcal{F}$ -closed by the obvious local argument.

*Homological tautness* complies *geometrical tautness*: take the form  $\omega$  just constructed using Hahn–Banach and homological tautness. Now construct point-wise projections  $\{P_\omega\}$  onto the tangent planes  $\{F\}$  of  $\mathcal{F}$  using purification. Happily, purification is natural and equal to the identity on  $(p+1)$  subspaces containing  $F$ . So the purified form  $\tilde{\omega}$  is *still* relatively  $\mathcal{F}$ -closed. Now construct any metric on the family of subspaces  $\{\text{kernel } P_\omega\}$  orthogonal direct sum any metric on the family of *tangent planes*  $\{F\}$  giving the volume forms  $\{\omega/F\}$ . For this metric  $\tilde{\omega}$  is the characteristic form, and by Rummeler’s calculation the leaves of  $\mathcal{F}$  are minimal surfaces.

*Stoke’s theorem implies the converse.* If  $c_n$  is a sequence of  $(p+1)$  chains tangent to  $\mathcal{F}$  so that  $\partial c_n$  approaches a foliation cycle  $z$  and  $\omega$  is a  $p$ -form positive of  $\mathcal{F}$  which is relatively  $\mathcal{F}$ -closed, we arrive at the contradiction

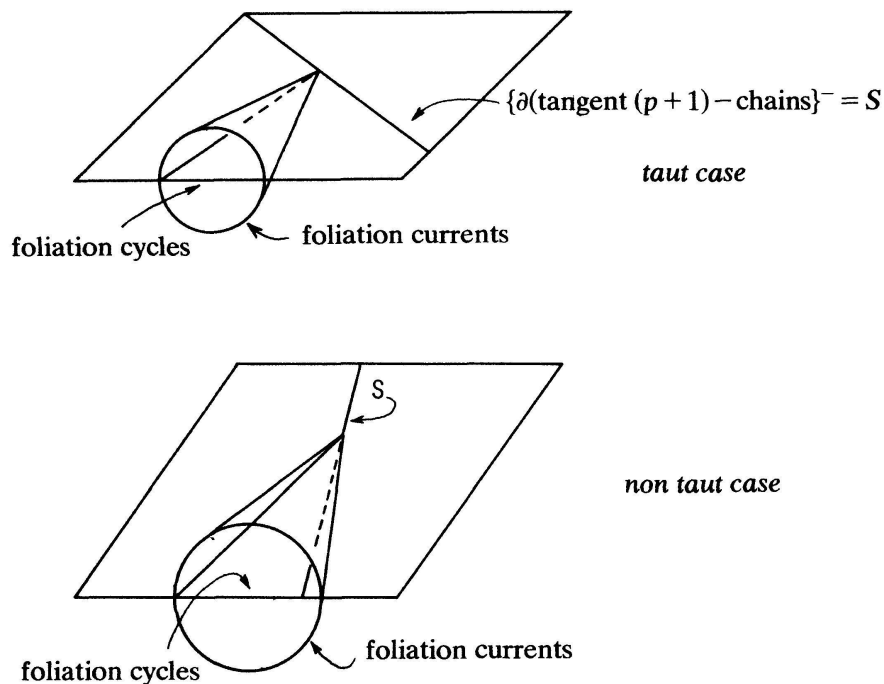
$$0 = \int_{c_n} d\omega = \int_{\partial c_n} \omega \rightarrow \int_z \omega > 0.$$

The proof of this theorem is now complete.

*Remark.* The proof shows that tautness of a foliation (geometrical or homological) is equivalent to either of the following conditions expressed by differential forms:

- (i) there is a  $p$ -form  $\omega$  positive on the oriented leaves of the foliation so that  $d\omega$  is zero on any  $(p+1)$  manifold tangent to the foliation.
- (ii) there is a pure  $p$ -form  $\omega$  satisfying the conditions of (i).

The equivalence of (ii) with geometrical tautness is Rummeler’s calculation [R] while the algebraic operation of purification  $\omega \rightarrow \tilde{\omega}$  shows (i) and (ii) are equivalent. Purification converts the *non-linear* problem (ii) into a *linear* problem (i) which may be treated as in  $[S_1]$  by Hahn–Banach. Thus we arrive at the necessary and sufficient homological condition of the theorem. The theorem is a generalization of the case  $\eta = 1$  treated in  $[S_2]$  which was in turn motivated by an interesting open letter from Hermann Gluck.



## EXAMPLES AND COROLLARIES

**COROLLARY 1.** *A foliation of a compact manifold has either a transversal invariant measure or for some Riemann metric all the leaves are minimal surfaces (of course, both can happen).*

**COROLLARY 2.** *A foliation is geometrically taut if no transversal invariant measure determines a trivial homology class.*

*Proof.* The boundaries form a closed subspace of currents.

**EXAMPLE.** Corollary 2 is illustrated by foliations which admit an immersed cross-section. (An immersed transversal submanifold cutting every leaf).

**COROLLARY 3.** *A codimension one oriented foliation is geometrically taut, if and only if every compact leaf is cut by a closed transversal curve.*

*Proof.* Transversal invariant measures carried on non-compact leaves intersect closed transversal curves and are thus determine foliation cycles essential in homology. By our assumption the same is true for the rest. (cf. Theorem II.20 [S<sub>1</sub>] where this curve condition was shown to be equivalent to the existence of a transversal volume preserving flow.)

**COROLLARY 4.** *Foliations by geodesics are characterized by the condition that no flow cycle approximately bounds a tangent homology. (cf. [S<sub>2</sub>]).*

Two general classes of foliations relevant to this discussion are “self-linking foliations” and “horospherical foliations.”

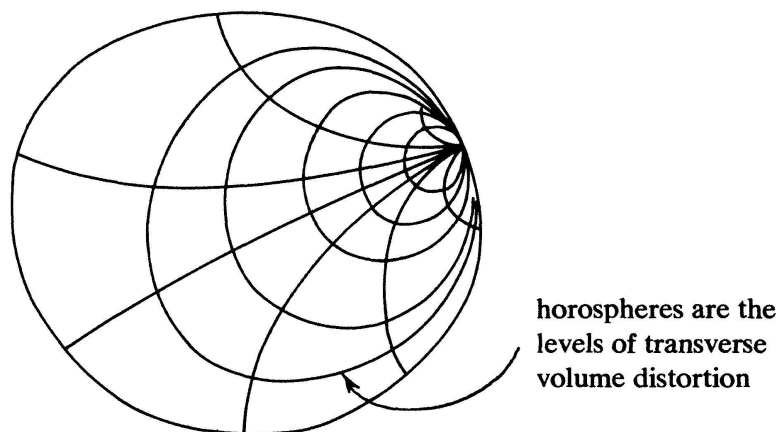
A “*self-linking foliation*” is by definition a  $p$ -dimensional foliation of a  $2p+1$  manifold defined by an *exact* pure  $(p+1)$ -form  $d\omega$  so that  $\omega \wedge d\omega$  is a *nowhere zero volume* form. Geometrically, such a foliation carries a diffuse foliation cycle (defined by  $d\omega$ ) which is homologous to zero ( $\omega$  is the homology) and the self-linking number ( $\omega \wedge d\omega$ ) is spread evenly over the entire manifold. When  $p = 1$  these self-linking foliations are just the contact flows.

**COROLLARY 5.** *Self-linking foliations are geometrically taut.*

*Proof.* Since  $\omega \wedge d\omega$  is a volume form,  $\omega$  is never zero on the leaves  $\{\text{kernel } d\omega\}$ . Clearly  $d\omega$  is zero on any manifold tangent to the foliation of dimension  $\leq 2p$  (in particular for  $p+1$ ). Thus  $\omega$  fulfills the condition of the Remark following the theorem. In fact, for any metric whose leaf volumes are  $\{\omega \mid \text{leaf}\}$  and whose orthogonal projection agree with  $\{P\omega\}$  the leaves are minimal surfaces.

Finally, non-taut examples can be constructed using the classical horospherical foliation as models. Define a “horospherical foliation” as one arising from another foliation of dimension one higher as the levels of distortion for a given transverse volume. More precisely, if  $\omega$  defines one foliation then  $d\omega = \eta \wedge \omega$  is pure and if it is nowhere zero defines a second foliation. This is the horospherical foliation associated to the first foliation defined by  $\omega$ . In the tangent bundles of negatively curved manifolds the first foliation is made of all geodesics asymptotic at  $\infty$  and the second is the foliation by horospheres.

**COROLLARY 6.** *The (generalized) horospherical foliations are never taut (geometrically or homologically).*



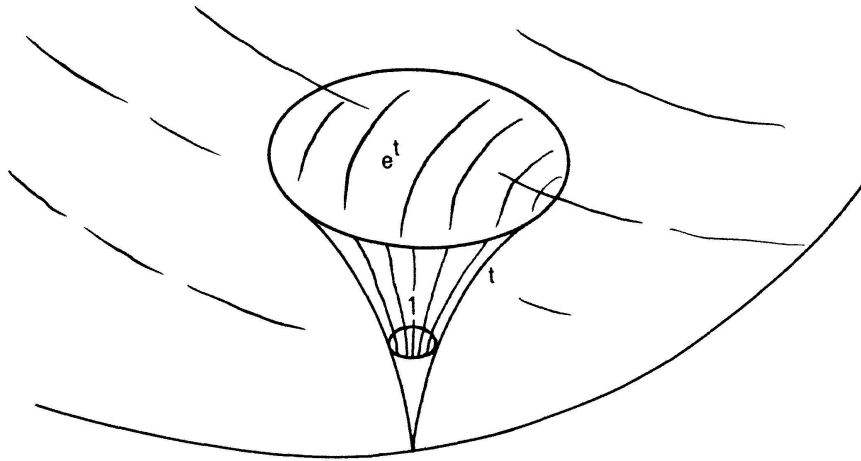
*Proof.* The diffuse foliation cycle defined by  $d\omega$  actually bounds an (infinite) tangent homology defined by  $\omega$ . This infinite homology may be approximated easily by a finite tangent homology  $[S_1]$  or one may simply use Stokes again. For example let  $\alpha$  be a form as in the Remark following the theorem. Then

$$0 < \int_M \alpha \wedge d\omega = \int_M d\alpha \wedge \omega = 0$$

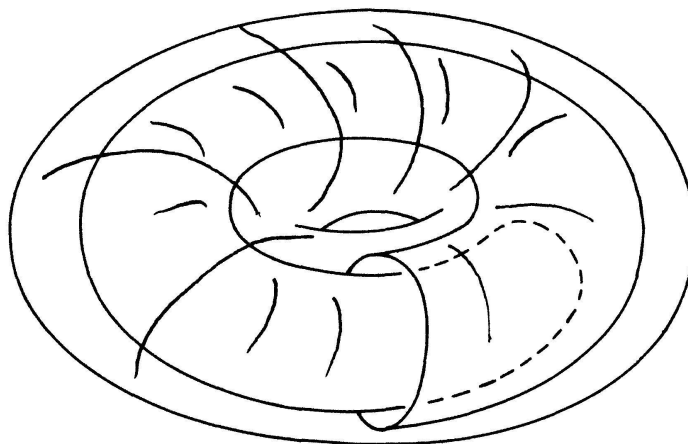
a contradiction.

## EXAMPLES OF TANGENT HOMOLOGIES

In the classical horospherical tangent examples there is a picture\* of the approximating homology defined by the region bounded by pieces of geodesics and horospheres of the indicated diameters.



Finite tangent homologies are easy to imagine. For example, Reeb components.



\* Contained in Plante's thesis for the upper half plane. (cf. [P]).

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