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The values of sectional curvature in indefinite metrics

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In his study of symmetric spaces with respect to *indefinite* metrics, J. Wolf has shown that the sign of the sectional curvature function of these manifolds is not well-behaved except in the case of manifolds of constant curvature. More specifically, he proves e.g. that if $M = G/H$ is a symmetric space where G is a simple Lie group, say and H is the connected component of the fixed point set of an involutive automorphism of G, and if the metric of M induced from that of G defined by the Killing form of the Lie algebra of G is indefinite, then the sectional curvature function of M is nonpositive iff it is bounded from above and below (cf. Wolf [1], theorem 1.3.1 or [2], theorem 12.1.5). Moreover, among the "isotropic" manifolds with indefinite metrics, Wolf shows that only the manifolds of constant curvature have bounded sectional curvature functions (cf. [1], theorems 2.9 and 4.1). If M has dimension 2, then this result follows easily from the fact that a connected 2-dimensional locally homogeneous space must have constant curvature, the sign of which may be fixed by requiring compatibility with an appropriate Killing form. On the other hand we show here that, for dimension ≥ 3, a more precise result holds without any hypothesis of homogeneity.

THEOREM. *Let M^n, $n \geq 3$, be a connected, smooth manifold with a smooth, indefinite metric. If the sectional curvature function of M is either bounded from above or bounded from below, then M is of constant curvature.*

Proof. Denote the sectional curvature function by K, the curvature tensor by R and the metric by $\langle\ ,\ \rangle$ and suppose that the index of the metric is h where $0 < h < n$. Recall that K is defined only for nondegenerate 2-dimensional subspaces: if v, w are linearly independent vectors at a point in M such that

$$\|v \wedge w\|^2 = \langle v, v\rangle\langle w, w\rangle - \langle v, w\rangle^2 \neq 0,$$

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and

$$K(\operatorname{span}\{v, w\}) = \frac{\langle R(v, w)v, w\rangle}{\|v \wedge w\|^2}.$$

Suppose that $K \neq$ constant. Then by Schur's theorem (cf. [2], 2.2.7), since $n \geq 3$, there exists a point p in M such that K does not take the same value for all nondegenerate 2-dimensional subspaces of the tangent space at p. The assertion is that at p itself K takes arbitrarily large positive and negative values. If this were false, then say K is bounded from below, say by d. (The argument is similar if K is bounded from above.) Consider a 3-dimensional subspace σ of the tangent space at p on which the metric is nondegenerate and *indefinite* and consider an orthonormal frame $\{e_i, e_j, e_\alpha\}$ spanning this subspace. For definiteness, let

$$\langle e_i, e_i\rangle = \langle e_j, e_j\rangle = -1, \qquad \langle e_\alpha, e_\alpha\rangle = 1. \tag{1.1}$$

Denote $\langle R(e_i, e_j)e_i, e_\alpha\rangle$ by $R_{iji\alpha}$ and $K(\operatorname{span}\{e_i, e_j\})$ by K_{ij}, etc. Then

$$K_{ij} = R_{ijij}, \qquad K_{i\alpha} = -R_{i\alpha i\alpha}.$$

By assumption,

$$K(\operatorname{span}\{xe_i + e_\alpha, e_j\}) = \frac{x^2K_{ij} + 2xR_{ij\alpha j} - K_{\alpha j}}{x^2 - 1} \geq d,$$

i.e.,

$$(x^2K_{ij} - K_{\alpha j}) + 2xR_{ij\alpha j} \begin{cases} \geq d(x^2-1) & \text{if} \quad |x| > 1 \\ \leq d(x^2-1) & \text{if} \quad |x| < 1 \end{cases} \tag{1.2}$$

so by continuity

$$x^2K_{ij} - K_{\alpha j} + 2xR_{ij\alpha j} = 0 \quad \text{if} \quad x = 1 \text{ or } -1.$$

This implies that

$$K_{ij} = K_{\alpha j}, \qquad R_{ij\alpha j} = 0 \tag{1.3}$$

and, by symmetry, also

$$K_{ij} = K_{\alpha i}, \qquad R_{i\alpha ij} = 0. \tag{1.3'}$$

Apply this result to the frame $\{xe_i + ye_\alpha, e_j, ye_i + xe_\alpha\}$ where x, y are arbitrary real numbers satisfying $x^2 - y^2 = 1$. It follows from (1.3) that

$$\langle R(xe_i + ye_\alpha, ye_i + xe_\alpha)xe_i + ye_\alpha, \qquad e_j\rangle = 0.$$

This easily leads to $R_{i\alpha j\alpha} = 0$. These conditions on the components of R suffice to imply that K takes the same value for all nondegenerate 2-dimensional subspaces contained in σ.

Let G(resp. $G(\sigma)$) be the Grassmannian of all unoriented 2-dimensional subspaces of the tangent space at p(resp. of σ) and let G_0(resp. $G_0(\sigma)$) be the subspace of G(resp. $G(\sigma)$) consisting of all *nondegenerate* 2-dimensional subspaces. Let S be the connected component of the identity of the group preserving a nonsingular form of index h on $\mathbf{R}^n$. Then S acts canonically on G and leaves G_0 invariant. The orbits of S in G_0 are easily seen to be the components of G_0. It follows that G_0 has two components if $h = 1$ or $n - 1$, and three components otherwise. Alternatively, a component of G_0 may be described as a subset consisting of all nondegenerate 2-dimensional, mutually isometric subspaces. For the same reason, $G_0(\sigma)$ has two components, one consisting of nondegenerate 2-dimensional subspaces on which the induced metric is *indefinite* and the other consisting of those subspaces on which the induced metric is *negative definite* (cf. the choice made in (1.1)). Now evidently K is an analytic function on G_0. A continuous variation of σ through 3-dimensional, nondegenerate subspaces shows that K is constant on an open subset intersecting two out of the possible three components of G_0 consisting of nondegenerate 2-dimensional subspaces on which the metric is indefinite or negative definite. By analyticity of K it follows that K is constant on these components. If G_0 has three components, then we apply the above reasoning to another family of nondegenerate, 3-dimensional subspaces σ' on which the induced metric has index 1. It follows that K takes the same value on all the components of G_0. The contradiction proves the theorem.

Q.E.D.

Remark 1. It follows from the above result that *the concepts of a manifold being "positively curved" or "negatively curved" based on the sign of K are vacuous for indefinite metrics and dimension* ≥ 3 *except in the case of constant curvature.*

Remark 2. It may be shown by means of examples that the sectional curvature function of a smooth manifold with a smooth indefinite metric and dimension ≥ 3 may be nonconstant and still may omit certain values.

REFERENCES

[1] WOLF, J.: Isotropic manifolds of indefinite metric, Commentarii Math. Helv. 39 (1964), 21–64.
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