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The invariance principle

MARTIN SCHECHTER

Abstract. We prove the invariance principle under weaker conditions and discuss applications.

1. Introduction

Let H_0, H be self-adjoint operators on Hilbert spaces $\mathcal{H}_0$, $\mathcal{H}$, respectively. We assume that there is a bounded linear map J from $\mathcal{H}_0$ to $\mathcal{H}$. Put

$$W(t)=e^{itH}Je^{-itH_0} \tag{1.1}$$

and let $M=M(H, H_0, J)$ be the set of those $f\in\mathcal{H}_0$ such that the limit

$$Wf=\lim_{t\to\infty} W(t)f \tag{1.2}$$

exists. It has been shown by several authors that under certain conditions on f and the real valued function $\varphi(s)$ that

$$Wf=\lim_{t\to\infty} e^{it\varphi(H)}Je^{it\varphi(H_0)f}f \tag{1.3}$$

(cf., e. g., Birman [1], Kato [2], Kato–Kuroda [3], Donaldson–Gibson–Hersh [4], Mateev [5], Sakhnovich [6], Matveev-Skriganov [7], Schechter [8], Chandler–Gibson [9]). When (1.3) holds, it is known as the invariance principle. The purpose of this paper is to prove it under conditions weaker than previously used. Applications are discussed in Section 5.

We shall make only one assumption on $\varphi(\zeta)$, namely that

$$\int_0^\infty \left|\int_I e^{-is\xi-it\varphi(\xi)}\,d\xi\right|^2 ds\to 0 \quad\text{as}\quad t\to\infty \tag{1.4}$$

for each bounded interval I. It was shown in [10] that (1.4) is satisfied if $\varphi'(s)$ is piecewise continuous, positive and locally of bounded variation. Our assumptions on f are also simpler and weaker than previously considered. Our theorems are stated in Sections 2, 6 and proved in Sections 4, 6. Lemmas which are used in their proofs are given in Section 3. We employ ideas from [3, 4, 8, 9].

2. The general theorems

Let $\mathcal{H}_{ac}(H)$ be the subspace of absolute continuity of H (for the definition cf. [10]). Our main theorems are

THEOREM 2.1. *Let $\varphi(s)$ be a real valued function satisfying* (1.4) *and let f be an element of $M \cap \mathcal{H}_{ac}(H_0)$ such that*

(2.1) *$d(E_0(s)f, f)/ds$ is bounded, E_0 and E are the spectral measures belonging to H_0 and H resp. and*

$$\int_0^\infty \|[W - W(s)]f\|^2 \, ds < \infty \tag{2.2}$$

Then $f \in M(\varphi(H), \varphi(H_0), J)$ and (1.3) *holds.*

THEOREM 2.2. *Let $\varphi(s)$ satisfy* (1.4) *and $f \in \mathcal{H}_{ac}(H_0)$ satisfy* (2.1). *Assume also that for some $p > 1$*

$$\int_1^\infty \|W'(t)f\|^p \, dt < \infty \tag{2.3}$$

and

$$\int_1^\infty \|W'(t)f\| (1 + \log t) \, dt < \infty \tag{2.4}$$

Then $f \in M(\varphi(H), \varphi(H_0), J)$ and (1.3) *holds.*

3. Some Lemmas

In this section we give some lemmas used in the proofs of Theorems 2.1 and 2.2.

LEMMA 3.1. *If* $u \in \mathcal{H}_{ac}(H)$ *and*

$$d(E(s)u, u)/ds \leq m^2 \tag{3.1}$$

then

$$\int |(e^{-itH}u, v)|^2 \, dt \leq 2\pi m^2 \|v\|^2 \tag{3.2}$$

Proof. Cf. [10].

LEMMA 3.2. *If* $u \in \mathcal{H}_{ac}(H)$, $v \in \mathcal{H}$ *then* $(E(s)u, v)$ *is absolutely continuous and*

$$|d/ds(E(s)u, v)|^2 \leq d/ds(E(s)u, u) d/ds(E(s)v_0, v_0)$$

where v_0 *is the projection of* v *onto* $\mathcal{H}_{ac}$.

Proof. Cf. [10, p. 517].

LEMMA 3.3. *If* (1.4) *holds, then*

$$\int_0^\infty \left\| \int e^{-i\xi s - it\varphi(\xi)} w(\xi) \, d\xi \right\|^2 ds \to 0 \quad \text{as} \quad t \to \infty \tag{3.3}$$

for each Bochner square integrable vector valued function $w(\xi)$.

Proof. Clearly (1.4) implies (3.3) for all step functions $w(\xi)$. But these functions are dense in the space of Bochner square integrable vector valued functions (cf [11]). Since the left hand side of (3.3) is bounded by

$$2\pi \int_{-\infty}^{\infty} \|w(\xi)\|^2 \, d\xi$$

the result follows.

LEMMA 3.4. *If* (1.4) *holds, then*

$$\int e^{-it\varphi(\xi)} h(\xi) \, d\xi \to 0 \quad \text{as} \quad t \to \infty \tag{3.4}$$

for all Bochner integrable vector valued functions $h(\xi)$.

Proof. Put $h_1(\xi)=\|h(\xi)\|^{\frac{1}{2}}$ when $h(\xi)\neq 0$ and $h_1(\xi)=e^{-\xi^2}$ when $h(\xi)=0$. Take $h_2(\xi)=h(\xi)/h_1(\xi)$. Then $h(\xi)=h_1(\xi)h_2(\xi)$ and h_1, h_2 are square integrable. Now the left hand side of (3.4) equals

$$\int w_{1,t}(s)w_{2,t}(s)^*\,ds \tag{3.5}$$

where

$$w_{1,t}(s)=(2\pi)^{-\frac{1}{2}}\int e^{-is\xi-\frac{1}{2}it\varphi(\xi)}h_1(\xi)\,d\xi$$

$$w_{2,t}(s)=(2\pi)^{-\frac{1}{2}}\int e_i^{\varphi s\xi+\frac{1}{2}it\varphi(\xi)}h_2(\xi)\,d\xi$$

But by Lemma 3.3

$$\int_0^\infty |w_{1,t}(s)|^2\,ds\to 0 \quad \text{as} \quad t\to\infty$$

and

$$\int_{-\infty}^0 \|w_{2,t}(s)\|^2\,ds\to 0 \quad \text{as} \quad t\to\infty$$

Thus the expression (3.5) tends to 0 as $t\to\infty$.

LEMMA 3.4. *Put*

$$g_t(\xi)=(2\pi)^{-1/2}\int_I e^{is\xi-it\varphi(s)}\,ds \tag{3.6}$$

$$U(t)=\int g_t(\xi)e^{-i\xi H}u(\xi)\,d\xi \tag{3.7}$$

and

$$G(s)=(2\pi)^{-1/2}\int_0^\infty e^{is\xi}u(\xi)\,d\xi \tag{3.8}$$

If $G(s)$ is Bochner integrable, then

$$\|U(t)\| \leq \int \|G(s)\|\, ds \tag{3.9}$$

and

$$U(t) \to 0 \quad \text{as} \quad t \to \infty \tag{3.10}$$

Proof. Cf. [9].

4. The proofs

Now we are ready for the proofs of the theorems of Section 2.

Proof of Theorem 2.1. Let $\epsilon > 0$ be given and take the bounded interval I so large that $\|E_0(CI)f\| < \epsilon$. Put

$$N(t) = (2\pi)^{-1/2}[W - J]e^{-itH_0} \tag{4.1}$$

Then

$$(N(t)f, v) = (2\pi)^{-1/2} \int e^{-its} h(s)\, ds$$

where

$$h(s) = d(E_0(s)f, [W^* - J^*]v)/ds \tag{4.3}$$

Define

$$(Z(t)f, v) = \int_I e^{-it\varphi(s)} h(s)\, ds \tag{4.4}$$

By (4.2), $(N(t)f, v)$ is the Fourier transform of $h(s)$. Hence

$$h(s) = (2\pi)^{-1/2} \int e^{-i\xi s} (N(\xi)f, v)\, d\xi \tag{4.5}$$

By Parseval's identity, (4.4) becomes

$$(Z(t)f, v) = \int g_t(\xi)(N(\xi)f, v)\, d\xi$$

where $g_t(\xi)$ is given by (3.6). Thus

$$Z(t)f = \int g_t(\xi)N(\xi)f\, d\xi \tag{4.6}$$

Put

$$Z_1(t)f = \int_0^\infty g_t(\xi)N(\xi)f\, d\xi \tag{4.7}$$

and

$$Z_2(t)f = \int_{-\infty}^0 g_t(\xi)N(\xi)f\, d\xi \tag{4.8}$$

Now

$$|(Z_2(t)f, v)|^2 \leqslant \int_{-\infty}^0 |g_t(\xi)|^2\, d\xi \int_{-\infty}^0 |(N(\xi)f, u)|^2\, d\xi$$

By Lemma 3.1, the last integral is bounded by $2\pi m^2 \|[W-J]^* v\|^2 \leqslant 2\pi m^2 \|W-J\|^2 \|v\|^2$. Thus

$$\|Z_2(t)f\|^2 \leqslant 2\pi m^2 \| W-J\|^2 \int_{-\infty}^0 |g_t(\xi)|^2\, d\xi \tag{4.9}$$

and this tends to 0 as $t \to \infty$ by (1.4) (note that the integral in (4.9) can be put in the form (1.4) by changing ξ to $-\xi$). Next we note that

$$Z_1(t)f = \int_I e^{-it\varphi(s)} w(s)\, ds \tag{4.10}$$

where

$$w(s) = (2\pi)^{-1/2} \int_0^\infty e^{is\xi} N(\xi)f\, d\xi \tag{4.11}$$

Since the limit (1.2) exists, we have

$$e^{isH}Wf = We^{isH_0}f \tag{4.12}$$

for each real s. Hence

$$(2\pi)^{1/2}e^{itH}N(t)f=[W-W(t)]f$$

and consequently

$$\|N(t)f\|=(2\pi)^{-1/2}\,\|[W-W(t)]f\|$$

Thus $N(t)f$ is square integrable in $(0,\infty)$ by (2.2). Hence $w(s)$ is in L^2 by (4.11). Since I is a bounded interval, $w(s)$ is integrable in I. If we now apply Lemma 3.4 to (4.10) we see that $Z_1(t)f\to 0$ as $t\to\infty$.

Next we note that by Lemma 3.2

$$\int_{CI} |h(s)|\,ds \leqslant \int_{CI}\left(\frac{d}{ds}\|E_0(s)f\|^2\right)^{1/2}\left(\frac{d}{ds}\|E_0(s)[W-J]^*v\|^2\right)^{1/2} ds$$

$$\leqslant \|E_0(CI)f\|\,\|\text{proj. of } (W-J)^*v \text{ onto } \mathscr{H}_{0ac}\|$$

Hence if we put

$$(Y(t)f,v)=\int_{CI} e^{-it\varphi(s)}h(s)\,ds$$

we will have

$$\|Y(t)f\|\leqslant 2\,\|E_0(CI)f\|<2\epsilon$$

Now

$$([W-J]e^{-it\varphi(H_0)}f,\,v)=\oint e^{-it\varphi(s)}h(s)\,ds$$

Hence

$$[W-J]e^{-it\varphi(H_0)}f=Y(t)f+Z(t)f$$

Since $\epsilon>0$ was arbitrary, we have

$$[W-J]e^{-it\varphi(H_0)}f\to 0 \quad \text{as} \quad t\to\infty$$

But

$$\|[W - e^{it\varphi(H)} J e^{-it\varphi(H_0)}]f\| = \|[W - J] e^{-it\varphi(H_0)} f\|$$

This gives (1.3).

Next we turn to the

Proof of Theorem 2.2. All of the hypotheses of Theorem 2.1 are assumed with the exception of (2.2), and this was used only to prove that

$$Z_1(t)f \to 0 \quad \text{as} \quad t \to \infty \tag{4.13}$$

Thus it suffices to prove (4.13). Let $\sigma(\xi)$ be a function which equals 1 for $\xi \geq 2$ and vanishes for $\xi \leq 1$. Put

$$Z_3(t)f = \int_0^\infty g_t(\xi)\sigma(\xi)N(\xi)f \, d\xi$$

and

$$Z_4(t)f = \int_0^\infty g_t(\xi)[1 - \sigma(\xi)]N(\xi)f \, d\xi$$

Then $Z_1 = Z_3 + Z_4$. Also

$$Z_4(t)f = \int_0 e^{-it\varphi(s)} M(s) \, ds$$

where

$$M(s) = (2\pi)^{-1/2} \int_0^\infty e^{is\xi}[1 - \sigma(\xi)]N(\xi)f \, d\xi$$

Since $1 - \sigma(\xi)$ vanishes for $\xi \geq 1$, we see that $[1 - \sigma(\xi)]N(\xi)f \in L^2$. Thus $M(s) \in L^2$ and consequently $M(s)$ is integrable on I. By Lemma 3.4 we see that $Z_4(t) \to 0$. Now

$$Z_3(t)f = \int_0^\infty g_t(\xi) e^{-i\xi H} u(\xi) \, d\xi$$

where

$$u(\xi)=\sigma(\xi)[W-W(\xi)]f$$

Thus

$$u'(\xi)=\sigma'(\xi)[W-W(\xi)]f-\sigma(\xi)W'(\xi)f$$

Since $\sigma'(\xi)$ vanishes for $\xi \geq 2$, (2.3) and (2.4) imply that $u' \in L^2$ and $[1+\log(1+|t|)]u' \in L^1$. Since $f \in M$ and $\sigma(0)=0$, we see that $u(0)=u(\infty)=0$. Let $G(s)$ be given by (3.8). Then

$$sG(s)=(2\pi)^{-1/2}i\int_0^\infty (e^{is\xi}-1)u'(\xi)\,d\xi$$

Thus

$$|s|\,\|G(s)\| \leq \int_0^\infty |\sin s\xi|\,\|u'(\xi)\|\,d\xi$$

and consequently

$$\int_{-1}^{1} \|G(s)\|\,ds \leq 2\int_0^\infty [1+\log(1+\xi)]\,\|u'(\xi)\|\,d\xi$$

Moreover, since

$$sG(s)=(2\pi)^{-1/2}i\int_0^\infty e^{is\xi}u'(\xi)\,d\xi$$

$$\begin{aligned}\int_1^\infty \|G(s)\|\,ds &\leq \left(\int_1^\infty s^{-p}\,ds\right)^{1/p}\left(\int_1^\infty \|sG(s)\|^{p'}\,ds\right)^{1/p'}\\ &\leq (p-1)^{-1}\left(\int \|u'(\xi)\|^p\,d\xi\right)^{1/p}\end{aligned}$$

Hence $G(s) \in L^1$. Now we can apply Lemma 3.4 to conclude that $Z_3(t)f \to 0$. Finally, we note that $f \in M$ by a well known theorem of Cook [12].

5. Applications

Condition (2.4) is not much stronger than a sufficient condition for the existence of the wave operator (1.2) due to Cook [12]. Condition (2.3) is close to it as well. Thus Theorem 2.2 states that the invariance principle holds under slightly stronger conditions than those usually used to prove the existence of the wave operators. As an illustration, let $H_0 = -\Delta$, $H = H_0 + V(x)$ on $L^2(E^n)$. For each $y \in E^n$ put

$$\psi_y(\xi) = \exp\{-|\xi|^2 - i\xi y\}$$

The linear combinations of such functions are dense in L^2, (2.1) holds and

$$|e^{-itH_0}\bar{F}\psi_y|^2 = (1+t^2)^{-n/2} \exp\left\{-\frac{|x-y|^2}{2(1+t^2)}\right\}$$

where F denotes the Fourier transform. Thus a sufficient condition for the existence of the wave operators is

$$\int_1^\infty (1+t^2)^{-n/4} T_y(t)\, dt < \infty \tag{5.2}$$

for each $y \in E^n$, where

$$T_y(t)^2 = \int |V(x)|^2 \exp\left\{-\frac{|x-y|^2}{2(1+t^2)}\right\} dx \tag{5.3}$$

Conditions (2.3) and (2.4) reduce to

$$\int_1^\infty (1+t^2)^{-pn/4} T_y(t)^p\, dt < \infty \tag{5.4}$$

and

$$\int_1^\infty (1+t^2)^{-n/4} \log(1+t) T_y(t)\, dt < \infty \tag{5.5}$$

respectively. Thus we have

THEOREM 5.1. *If $\varphi(s)$ satisfies* (1.4), *and* (5.4), (5.5) *hold, then the wave operators exist and the invariance principle holds.*

In particular, (5.4) and (5.5) both hold if

$$T_y(t) = 0(t^{-\alpha}) \quad \text{as} \quad t \to \infty \tag{5.6}$$

for some $\alpha > 1 - \frac{1}{2}n$ for each y. Thus we have

COROLLARY 5.2. *If φ satisfies* (1.4) *and V satisfies* (5.6), *then the invariance principle holds.*

Finally we note that (5.6) is implied by

$$(1+|x|)^{\alpha} V(x) \in L^2 \tag{5.7}$$

for some $\alpha > 1 - \frac{1}{2}n$. This is the condition for existence of the wave operators derived by Kuroda [13]. We have shown that it is also sufficient for the invariance principle to hold.

6. Another approach

Now we consider hypotheses that differ from those of Section 2. We shall assume that there are a Hilbert space $\mathscr{K}$ and closed operators A from $\mathscr{H}_0$ to $\mathscr{K}$ and B from $\mathscr{H}$ to $\mathscr{K}$ such that

$$(W'(t)f, g) = (Ae^{-itH_0}f, Be^{-itH}g)\mathscr{K} \tag{6.1}$$

We shall prove

THEOREM 6.1. *Assume that* (1.4) *and* (6.1) *hold. Let $f \in M \cap \mathscr{H}_{ac}(H_0)$ be such that*

$$\int_0^{\infty} (\|Ae^{-itH_0}f\|^2_{\mathscr{K}} + \|BWe^{-itH_0}f\|^2_{\mathscr{K}})\, dt < \infty \tag{6.2}$$

and

$$\limsup_{t\to\infty} \|Je^{-it\varphi(H_0)}f\| \leq \|Wf\| \tag{6.3}$$

Then $f \in M(\varphi(H), \varphi(H_0), J)$ and (1.3) *holds.*

Proof. By (6.1) we have

$$[(W-J]f, g)=\int_0^{\infty}(Ae^{-itH_0}f, Be^{-itH}g)\, ds \tag{6.4}$$

Now

$$(A[R_0(z)-R_0(\bar{z})]f, g)=-2\pi i\int \delta_a(\lambda-s)\, d(AE_0(\lambda)f, g)\to$$

$$-2\pi i\, d(AE_0(s)f, g)/ds \quad \text{a.e.} \quad \text{as } a\downarrow 0$$

where $z=s+ia$, $a>0$ and $\delta_a(\mu)=a/\pi(\mu^2+a^2)$. On the other hand

$$\int \|A[R_0(z)-R_0(\bar{z})]f-f(s)\|^2\, ds\to 0 \quad \text{as} \quad a\to 0$$

where

$$f(s)=-i\int_{-\infty}^{\infty} e^{ist}Ae^{-itH_0}f\, dt \tag{6.5}$$

Hence

$$2\pi d(AE_0(s)f, g)/ds=i(f(s), g) \quad \text{a.e.} \tag{6.6}$$

Consequently, if $\alpha(\lambda)$ is any Borel function, we have

$$\begin{aligned}(A\alpha(H_0)f, g)&=\int \alpha(\lambda)d(AE_0(\lambda)f, g)\\ &=i(2\pi)^{-1}\int \alpha(\lambda)(f(\lambda), g)\, d\lambda\end{aligned}$$

or

$$A\alpha(H_0)f=i(2\pi)^{-1}\int \alpha(\lambda)f(\lambda)\, d\lambda \tag{6.7}$$

In particular, we have

$$\int_0^{\infty}\|Ae^{-i\sigma H_0-it\varphi(H_0)}f\|^2\, d\sigma=$$

$$(2\pi)^{-2}\int_0^{\infty}\left\|\int e^{-i\sigma\lambda-it\varphi(\lambda)}f(\lambda)\, d\lambda\right\|^2 d\sigma\to 0 \quad \text{as} \quad t\leftarrow\infty$$

by Lemma 3.3. Similarly,

$$\int \|BW[R_0(\bar{z})-R_0(z)]f-h(s)\|^2\,ds\to 0 \quad \text{as} \quad a\to 0$$

where

$$h(s)=-i\int_{-\infty}^{\infty} e^{ist}BWe^{-itH_0}f\,dt$$

Hence

$$\int_0^{\infty}\|Be^{-i\sigma H-it\varphi(H)}Wf\|^2\,d\sigma=(2\pi)^{-2}\int_0^{\infty}\left\|\int e^{-i\sigma\lambda-it\varphi(\lambda)}h(\lambda)\,d\lambda\right\|^2 d\sigma$$

$$=(2\pi)^{-2}\int_0^{\infty}\left\|\int e^{-i\sigma\lambda-it\varphi(\lambda)}\,h(\lambda)\,d\lambda\right\|^2 d\sigma$$

$$\leq(2\pi)^{-1}\int\|h(\lambda)\|^2\,d\lambda=\int\|BWe^{-i\sigma H_0}f\|^2\,d\sigma$$

Now by (6.4)

$$|([W-J]e^{-it\varphi(H_0)}f,\,e^{-it\varphi(H)}Wf)|^2\leq\int_0^{\infty}\|Ae^{-i\sigma H_0-it\varphi(H_0)}f\|^2\,d\sigma$$

$$\times\int_0^{\infty}\|Be^{-i\sigma H-it\varphi(H)}Wf\|^2\,d\sigma$$

The first integral converges to 0 as $t\to\infty$ while the second is bounded. Thus

$$(Wf,\,e^{it\varphi(H)}Je^{-it\varphi(H_0)}f)\to\|Wf\|^2 \quad \text{as} \quad t\to\infty \tag{6.8}$$

Hence

$$\|[e^{it\varphi(H)}Je^{-it\varphi(H_0)}-W]f\|^2=\|Je^{-it\varphi(H_0)}f\|^2$$
$$+\|Wf\|^2-2\,\mathrm{Re}\,(e^{it\varphi(H)}Je^{-it\varphi(H_0)}f,\,Wf) \tag{6.9}$$

By (6.3) and (6.8), the right hand side of (6.9) converges to 0. This gives the desired result.

It should be noted that (6.3) holds if $[J^*J-W^*W]E_0(I)$ is a compact operator for each bounded interval I. To see this, let $\epsilon>0$ be given and take I solarge that

$\|E_0(CI)f\| < \epsilon$. Now $e^{-it\varphi(H_0)}E_0(I)f$ converges to 0 weakly since

$$(e^{-it\varphi(H_0)}E_0(I)f, g) = \int_I e^{-it\varphi(\lambda)}\frac{d}{d\lambda}(E_0(\lambda)f, g)\, d\lambda$$

This converges to 0 by Lemma 3.4. Thus $[J^*J - W^*W]e^{-it\vartheta(H_0)}E_0(I)f$ converges strongly to 0. Hence

$$\|Je^{-it\varphi(H_0)}E_0(I)f\|^2 - \|WE_0(I)f\|^2 = \\ ([J^*J - W^0W]e^{-it\varphi(H_0)}E_0(I)f, e^{-it\varphi(H_0)}E_0(I)f) \to 0$$

On the other hand

$$\|Je^{-it\varphi(H_0)}E_0(CI)f\| < \|J\|\epsilon$$

$$\|WE_0(CI)f\| < \|W\|\epsilon$$

Thus

$$\|Je^{-it\varphi(H_0)}f\| \to \|Wf\| \quad \text{as} \quad t \to \infty$$

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Dept. of Mathematics, Yeshiva University
500 West 185th Street
New York, N.Y. 10033 U.S.A.

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