On the first eigenvalue of the Laplacian for compact submanifolds of Euclidean space.

Comment. Math. Helvetici **52** (1977) 525–533 Birkhäuser Verlag, Basel

On the first eigenvalue of the Laplacian for compact submanifolds of Euclidean space

ROBERT C. REILLY*

1. Introduction

This paper was inspired by recent work of D. Bleecker and J. Weiner [3]. The following results are typical examples of those we obtain in this paper.

THEOREM. *The first eigenvalue of the Laplacian for a compact n-manifold isometrically immersed in Euclidean space is bounded above by n times the average value of the square of the norm of the mean curvature vector. Moreover, if the eigenvalue achieves this bound, then the submanifold is actually a minimal submanifold of some hypersphere in the Euclidean space. (See the case $r = 1$ in Theorem A.)*

COROLLARY. *If a compact connected n-manifold is immersed as a hypersurface in $\mathbf{R}^{n+1}$ so as to have the same constant mean curvature and same first eigenvalue as an n-sphere of radius R, then it is immersed as an n-sphere of radius R.*

Among our results are included generalizations of Theorems I and II of [3].

2. Notation and preliminary results

Throughout this paper M denotes a smooth (that is, C^∞), compact, oriented connected n-dimensional manifold without boundary and $\mathbf{Y}$ denotes a smooth immersion of M into Euclidean space $\mathbf{R}^{n+p}$. We always assume that M is endowed with the Riemannian structure induced by $\mathbf{Y}$ from the inner product $\langle , \rangle$ on $\mathbf{R}^{n+p}$. We denote the volume form, volume and Laplace–Beltrami operator on M

* Research for this paper was completed while the author was on sabbatical leave from the University of California (Irvine) and in residence at the Mathematics Institute of the University of Warwick.

(defined relative to this Riemannian structure) by dA, A and Δ (respectively). We denote the second fundamental form, which is normal-vector valued, by $\mathbf{B}$, and its matrix relative to an orthonormal frame $\mathbf{e}_1, \ldots, \mathbf{e}_n$ by $(\mathbf{b}_{ij})$. If $\mathbf{u}$ is a unit normal vector at a point of M, we denote the (real valued) second fundamental form $\langle \mathbf{B}, \mathbf{u} \rangle$ along $\mathbf{u}$ by $B^{\mathbf{u}}$. We denote the square of the length of $\mathbf{B}$, which can be computed as $\sum_{i,j} \langle \mathbf{b}_{ij}, \mathbf{b}_{ij} \rangle$, by $|\mathbf{B}|^2$. Similarly, we set $|B^{\mathbf{u}}|^2 = \sum_{ij} \langle \mathbf{b}_{ij}, \mathbf{u} \rangle \langle \mathbf{b}_{ij}, \mathbf{u} \rangle$. These quantities do not depend on the choice of orthonormal frame. If $p = 1$, we denote the unit normal to M (determined by the orientation on M) by $\mathbf{N}$, and we denote the support function $\langle \mathbf{Y}, \mathbf{N} \rangle$ by P. Next we define the mean curvatures and prove the Hsiung–Minkowski formulas for arbitrary codimension p. (We have already done this in an earlier paper [6], but the book in which it appears does not seem to be readily available. Our derivation of the formulas here is simpler than that in [6].)

DEFINITION. If r is an integer, $0 \leq r \leq n$, then the r-th mean curvature on M is the quantity

$$\binom{n}{r}^{-1} \frac{1}{r!} \sum \varepsilon(i_1, \ldots, i_r; j_1, \ldots, j_r) \langle \mathbf{b}_{i_1 j_1}, \mathbf{b}_{i_2 j_2} \rangle \cdots \langle \mathbf{b}_{i_{r-1} j_{r-1}}, \mathbf{b}_{i_r j_r} \rangle \text{ if } r \text{ is even,}$$

$$\binom{n}{r}^{-1} \frac{1}{r!} \sum \varepsilon(i_1, \ldots, i_r; j_1, \ldots, j_r) \langle \mathbf{b}_{i_1 j_1}, \mathbf{b}_{i_2 j_2} \rangle \cdots \langle \mathbf{b}_{i_{r-2} j_{r-2}}, \mathbf{b}_{i_{r-1} j_{r-1}} \rangle \mathbf{b}_{i_r j_r}, \text{ if } r \text{ is odd.}$$

Here the ε's are the usual permutation symbols, and the sums are taken over all values of the indices from 1 to n. We set $\sigma_0 = 1$.

Note that if r is odd, then the r-th mean curvature is normal-vector valued; we denote it by $\boldsymbol{\sigma}_r$. For example, $\boldsymbol{\sigma}_1 = (\sum_j \mathbf{b}_{jj})/n$. In contrast, if r is even, then the r-th mean curvature is real-valued; we denote it by σ_r. One can readily show, by using the skew-symmetries of the permutation symbols, together with the Gauss curvature equations, that when r is even, σ_r is a polynomial in the components of the curvature tensor. For example, $n(n-1)\sigma_2$ is the scalar curvature.

When the codimension p is 1, it is convenient to define real valued mean curvatures of odd order by the rule $\sigma_r = \langle \boldsymbol{\sigma}_r, \mathbf{N} \rangle$. We also set $\sigma_{-1} = -P$.

PROPOSITION 1. (a) *If $p > 1$ and r is an odd integer, $1 \leq r \leq n$, then*

$$\int_M (\langle \mathbf{Y}, \boldsymbol{\sigma}_r \rangle + \sigma_{r-1}) \, dA = 0 \tag{1}$$

(b) *If $p=1$ and r is any integer, $0 \leq r \leq n$, then*

$$\int_M (\langle \mathbf{Y}, \sigma_r \mathbf{N} \rangle + \sigma_{r-1})\, dA = 0. \tag{2}$$

Proof. Essentially the same proof works for either case; since (b) is proved in [4], we'll do only (a).

Consider a tangent vector field $\mathbf{X}$ on M whose components $X_1, \ldots, X_n$ relative to an orthonormal frame field $\mathbf{e}_1, \ldots, \mathbf{e}_n$ are given by

$$X_j = \frac{1}{(r-1)!} \sum \varepsilon\,(i_1, \ldots, i_{r-1}, i; j_1, \ldots, j_{r-1}, j) \langle \mathbf{b}_{i_1 j_1}, \mathbf{b}_{i_2 j_2} \rangle \cdots \langle \mathbf{b}_{i_{r-2} j_{r-2}}, \mathbf{b}_{i_{r-1} j_{r-1}} \rangle \langle \mathbf{Y}, \mathbf{e}_i \rangle.$$

(The sum is over the indices $i_1, \ldots, i_{r-1}, i$ and $j_1, \ldots, j_{r-1}$.) (It is easy to check that this formula for $\mathbf{X}$, like those for the mean curvatures, does not depend on the choice of frame field.) One readily checks, using the Codazzi equations $\mathbf{b}_{ij,k} = \mathbf{b}_{ik,j}$ (where the comma denotes covariant differentiation in the normal bundle) and the skew-symmetries of the permutation symbols, that $\operatorname{div} \mathbf{X}$ is a constant multiple of the integrand in (1). Then Stokes' theorem yields the result.

The next three propositions are well known; we state them, without proof, for future reference.

PROPOSITION 2. (*Minimum principle*) *If λ_1 is the smallest positive eigenvalue of the Laplace–Beltrami operator Δ on M and if $f: M \to \mathbf{R}$ is a C^1 function such that $\int_M f\, dA = 0$ then $\int_M |\operatorname{grad} f|^2\, dA \geq \lambda_1 \int_M f^2\, dA$; equality implies $\Delta f = -\lambda_1 f$. (See* [1].)

PROPOSITION 3. (*Averaging principle*) *Let S^{N-1} be the unit sphere in Euclidean space $\mathbf{R}^N$ and let $d\mathbf{X}$ denote the standard ($SO(N)$-invariant) volume element on S^{N-1}, normalized so that $\int_{S^{N-1}} 1 \cdot d\mathbf{X} = 1$.*

If $\mathbf{B}, \mathbf{C}$ are any vectors in $\mathbf{R}^N$, then

$$\int_{S^{N-1}} \langle \mathbf{B}, \mathbf{X} \rangle \langle \mathbf{C}, \mathbf{X} \rangle\, d\mathbf{X} = (1/N) \langle \mathbf{B}, \mathbf{C} \rangle.$$

PROPOSITION 4. (*Newton's Inequality*) *If (a_{ij}) is a symmetric $n \times n$ real matrix then $\sum_{i,j} a_{ij}^2 \geq 1/n (\sum_i a_{ii})^2$; moreover, we get equality if and only if (a_{ij}) is proportional to the identity matrix.*

3. Upper bounds for λ_1

All of our results flow from a simple lemma.

MAIN LEMMA. *If* $\mathbf{Y}: M \to \mathbf{R}^{n+p}$ *is an immersion for which* $\int_M \mathbf{Y}\, dA = \mathbf{0}$, *then*

$$nA \geq \lambda_1 \int_M |\mathbf{Y}|^2\, dA. \tag{3}$$

Equality in (3) implies $\mathbf{Y}$ is a minimal immersion of M into a hypersphere of $\mathbf{R}^{n+p}$.

Proof. We restate the hypothesis as: for any unit vector $\mathbf{X} \in \mathbf{R}^{n+p}$, $\int_M \langle \mathbf{Y}, \mathbf{X}\rangle\, dA = 0$.

Thus, for any such $\mathbf{X}$, the function $f = \langle \mathbf{Y}, \mathbf{X}\rangle : M \to \mathbf{R}$ satisfies the hypotheses of Proposition 2, so we can assert that

$$\int_M |\text{grad}\, \langle \mathbf{Y}, \mathbf{X}\rangle|^2\, dA \geq \lambda_1 \int_M \langle \mathbf{Y}, \mathbf{X}\rangle^2\, dA. \tag{4}$$

Now if $\mathbf{e}_1, \ldots, \mathbf{e}_n$ is an orthonormal frame field at a point $q \in M$, one easily checks that, at q, $|\text{grad}\, \langle \mathbf{Y}, \mathbf{X}\rangle|^2 = \sum_i \langle \mathbf{e}_i, \mathbf{X}\rangle^2$. Thus, by Proposition 3 we have (at q)

$$\int_{S^{n+p-1}} |\text{grad}\, \langle \mathbf{Y}, \mathbf{X}\rangle|^2\, d\mathbf{X} = \sum_i \int_{S^{n+p-1}} \langle \mathbf{e}_i, \mathbf{X}\rangle^2\, d\mathbf{X} = \sum_i \langle \mathbf{e}_i, \mathbf{e}_i\rangle/(n+p) = n/(n+p).$$

Similarly, we have (at q) $\int_{S^{n+p-1}} \langle \mathbf{Y}, \mathbf{X}\rangle^2\, d\mathbf{X} = |\mathbf{Y}|^2/(n+p)$. Then if we integrate both sides of (4) with respect to $\mathbf{X}$, switch the order of the integrations (allowable by Fubini's theorem) and multiply both sides by $(n+p)$, we obtain (3). Equality in (3) implies (by Proposition 2) that, for each $\mathbf{X}$, $\Delta\langle \mathbf{Y}, \mathbf{X}\rangle = -\lambda_1\langle \mathbf{Y}, \mathbf{X}\rangle$. By Takahashi's Theorem ([7, Thm. 3]) this implies that $\mathbf{Y}$ minimally immerses M into a hypersphere.

Let us now state our main results. We continue with the notation of the preceding section.

THEOREM A. (a) *If* $p > 1$ *and* r *is an odd integer,* $1 \leq r \leq n$, *then*

$$nA \int_M |\boldsymbol{\sigma}_r|^2\, dA \geq \lambda_1 \left(\int_M \sigma_{r-1}\, dA\right)^2. \tag{5}$$

If, for some such r, we have equality in (5) *and if* $\boldsymbol{\sigma}_r$ *does not vanish identically, then* $\mathbf{Y}$ *immerses M minimally into some hypersphere in* $\mathbf{R}^{n+p}$ *and* $\boldsymbol{\sigma}_r$ *is parallel in the normal bundle of M in* $\mathbf{R}^{n+p}$. *In particular, if* $r = 1$ *and we have equality in* (5), *then* $\mathbf{Y}$ *immerses M as a minimal submanifold of some hypersphere in* $\mathbf{R}^{n+p}$. (*We don't have to assume that* $\boldsymbol{\sigma}_1$ *does not vanish identically.*)

(b) *If* $p = 1$ *and r is any integer,* $0 \leq r \leq n$, *then*

$$nA \int_M \sigma_r^2 \, dA \geq \lambda_1 \left(\int_M \sigma_{r-1} \, dA \right)^2. \tag{6}$$

We get equality in (6) for some r, $0 \leq r \leq n$, if and only if $\mathbf{Y}$ immerses M as a hypersphere in $\mathbf{R}^{n+1}$. If, in addition, $n \geq 2$, $\mathbf{Y}$ will be an imbedding.

Proof. Since all the quantities appearing in (5) and (6) are independent of the choice of origin, we may, without loss of generality, assume that the center of gravity of $\mathbf{Y}$ is located at the origin; that is, $\int_M \mathbf{Y} \, dA = \mathbf{0}$. (The only expression that appears to depend on the choice of origin is $\int_M P \, dA$, which shows up in (6) when $r = 0$ (recall that $\sigma_{-1} = -P$). In fact, it does not depend on that choice. For if we translate $\mathbf{Y}$ by a constant vector $\mathbf{C}$, then the new support function is $P' = \langle \mathbf{Y} + \mathbf{C}, \mathbf{N} \rangle = P + \langle \mathbf{C}, \mathbf{N} \rangle$. However, it is well known that for a compact hypersurface in $\mathbf{R}^{n+1}$, $\int_M \mathbf{N} \, dA = \mathbf{0}$). With this assumption in force, we can apply the main lemma, i.e., (3) holds.

To prove (a), first multiply both sides of (3) by the quantity $\int_M |\boldsymbol{\sigma}_r|^2 \, dA$. Then, after using the Cauchy–Schwarz inequalities for integrals and for vectors in $\mathbf{R}^{n+p}$, we obtain the following string of inequalities:

$$\begin{aligned} nA \int_M |\boldsymbol{\sigma}_r|^2 \, dA &\geq \lambda_1 \left(\int_M |\mathbf{Y}|^2 \, dA \right) \left(\int_M |\boldsymbol{\sigma}_r|^2 \, dA \right) \\ &\geq \lambda_1 \left(\int_M |\mathbf{Y}| \, |\boldsymbol{\sigma}_r| \, dA \right)^2 \geq \lambda_1 \left(\int_M \langle \mathbf{Y}, \boldsymbol{\sigma}_r \rangle \, dA \right)^2. \end{aligned} \tag{7}$$

We get the desired inequality (5) by applying the Hsiung–Minkowski formula (1) to the rightmost integral in (7).

If we get equality in (5), then all the inequalities in (7) must in fact be equations. By the basic facts for Cauchy–Schwarz inequalities, this implies that $\boldsymbol{\sigma}_r = C \cdot \mathbf{Y}$ for some constant C. Since, by hypothesis, $\boldsymbol{\sigma}_r$ does not vanish identically, we must have $C \neq 0$. Thus, since $\boldsymbol{\sigma}_r$ is normal-vector valued, $\mathbf{Y}$ is always normal to M. Since $d\,|\mathbf{Y}|^2 = 2\langle \mathbf{Y}, d\mathbf{Y} \rangle = 0$, it follows that $|\mathbf{Y}|$ is constant, so $\mathbf{Y}$ maps M into a hypersphere of $\mathbf{R}^{n+p}$. Similarly, since $d\mathbf{Y}$ is tangent to M and $\boldsymbol{\sigma}_r = C\mathbf{Y}$, we see that $\boldsymbol{\sigma}_r$ is parallel in the normal bundle. In addition, equality in (7) implies equality in (3) which, by the final part of the main lemma, implies that $\mathbf{Y}$ minimally immerses M into a hypersphere. If $r=1$ then $\boldsymbol{\sigma}_1$ cannot vanish identically, since (by (1)), $\int_M \langle \mathbf{Y}, \boldsymbol{\sigma}_1\rangle\, dA = -A \neq 0$.

The proof of (b) follows in much the same way, so we'll omit it, except to note that when $r=0$ we use the inequalities

$$|\mathbf{Y}|\sigma_0 = |\mathbf{Y}|\cdot 1 = |\mathbf{Y}|\cdot|\mathbf{N}| \geq |\langle \mathbf{Y}, \mathbf{N}\rangle| = |P|.$$

COROLLARY 1. *If $p=1$ and $\mathbf{Y}$ imbeds M as the boundary of the domain $D\subset \mathbf{R}^{n+1}$, then $nA^2 \geq \lambda_1(n+1)^2V^2$, where V is the $(n+1)$-dimensional volume of D. Moreover, we get equality if and only if D is a ball.*

Proof. Apply (b) of Theorem A in the case $r=0$, after observing that, by Stokes' theorem, $|\int_M P\, dA| = \int_D (n+1)\, dV = (n+1)V$.

COROLLARY 2. *If, in the preceding Corollary, we further assume that $n=1$, then we obtain the classical isoperimetric inequality in the plane: if A is the length of the boundary of a domain D in $\mathbf{R}^2$ and V is the area of D, then $A^2 \geq 4\pi V$, with equality if and only if the domain is a disc.*

Proof. Use the inequality provided by Corollary 1 when $n=1$ and recall that for a closed curve of length A the first eigenvalue can be computed explicitly: $\lambda_1 = (2\pi/A)^2$.

Remark. Our proof of Theorem A is nothing but a simple modification and generalization of the well known proof of Corollary 2 due to A. Hurwitz (1902). (For a discussion of Hurwitz' proof, see [2, esp. pp. 43–45].)

THEOREM B. *Using the notation of Section 2, we have the following inequality:*

$$\int_M |\mathbf{B}|^2\, dA \geq \lambda_1 A. \tag{8}$$

Moreover, we have equality if and only if $\mathbf{Y}$ immerses M as a hypersphere in some $(n+1)$-dimensional linear subspace of $\mathbf{R}^{n+p}$.

Proof. By Newton's inequality (Proposition 4) one sees that if $\mathbf{u}$ is any unit normal vector (at a given point of M) then $|B^{\mathbf{u}}|^2 \geq n\langle \boldsymbol{\sigma}_1, \mathbf{u}\rangle^2$, with equality if and only if $B^{\mathbf{u}}$ is proportional to the identity (i.e., if and only if "M is umbilic in the normal direction $\mathbf{u}$"). If we integrate this inequality over all such unit normals $\mathbf{u}$, Proposition 3 implies $|\mathbf{B}|^2 \geq n\,|\boldsymbol{\sigma}_1|^2$, with equality if and only if each $B^{\mathbf{u}}$ is proportional to the identity. Thus, if we integrate over M we get

$$\int_M |\mathbf{B}|^2\,dA \geq \int_M n\,|\boldsymbol{\sigma}_1|^2\,dA \tag{9}$$

which, together with (5) (in the case $r=1$) implies (8). Moreover, if we have equality in (8), we must also have it in (9), and thus, by the preceding remarks, $\mathbf{Y}$ must be umbilic in all normal directions and at all points. It is well known that this implies that $\mathbf{Y}$ maps M into a hypersphere of some $(n+1)$-dimensional linear subspace of $\mathbf{R}^{n+p}$.

THEOREM C. *If $\boldsymbol{\sigma}_1$ is never $\mathbf{0}$, then*

$$\int_M |B^{\mathbf{u}}|^2\,dA \geq \lambda_1 A \tag{10}$$

where $\mathbf{u} = \boldsymbol{\sigma}_1/|\boldsymbol{\sigma}_1|$ is the unit normal field in the direction of $\boldsymbol{\sigma}_1$. Moreover, if we have equality in (10) then $\mathbf{Y}$ immerses M as a minimal submanifold of some hypersphere in $\mathbf{R}^{n+p}$.

Proof. Apply Theorem A and Proposition 4.

Remarks. Theorem B is the same as Theorem I of [3]; however, because of the availability of Theorem A, our proof of the second part of Theorem B is easier than that in [3]. Theorem C is a strengthening of Theorem II of [3]; Bleecker and Weiner require the additional hypothesis that $\boldsymbol{\sigma}_1$ be parallel in the normal bundle. The following result generalizes their proof of Theorem II in [3].

THEOREM D. *Suppose that for some odd r, $1 \leq r \leq n$, $\boldsymbol{\sigma}_r$ is parallel in the normal bundle and is nonzero. Let $\mathbf{u} = \boldsymbol{\sigma}_r/|\boldsymbol{\sigma}_r|$ be the unit normal field in the direction of $\boldsymbol{\sigma}_r$. Then we have the following inequality:*

$$\int_M |B^{\mathbf{u}}|^2\,dA \geq \lambda_1 A. \tag{11}$$

Proof. The Hsiung–Minkowski formula (1) holds, independently of the choice of origin. Thus if we replace $\mathbf{Y}$ by $\mathbf{Y}+\mathbf{X}$ in (1), where $\mathbf{X}$ is any unit vector in $\mathbf{R}^{n+p}$, we still have a valid formula. It follows that for all such $\mathbf{X}$, $\int_M \langle \mathbf{X}, \boldsymbol{\sigma}_r\rangle\,dA = 0$.

Thus we may apply Proposition 2 to the function $f = \langle \mathbf{X}, \boldsymbol{\sigma}_r \rangle$, which we can also write as $f = \langle \mathbf{X}^\perp, \boldsymbol{\sigma}_r \rangle$, where $\mathbf{X}^\perp$ is the normal component of $\mathbf{X}$. Then one readily checks (using covariant differentiation in the normal bundle on $\mathbf{X}^\perp$ and $\boldsymbol{\sigma}_r$) that, because $\boldsymbol{\sigma}_r$ is parallel,

$$|\text{grad } f|^2 = |\boldsymbol{\sigma}_r|^2 \sum_{ijk} \langle \mathbf{X}, \mathbf{e}_i \rangle \langle \mathbf{X}, \mathbf{e}_k \rangle b_{ij}^{\mathbf{u}} b_{kj}^{\mathbf{u}}.$$

(We have introduced the orthonormal frame field $\mathbf{e}_i, \ldots, \mathbf{e}_n$ for these calculations.) Thus Proposition 2 implies that

$$\int_M |\boldsymbol{\sigma}_r|^2 \left(\sum_{ijk} \langle \mathbf{X}, \mathbf{e}_i \rangle \langle \mathbf{X}, \mathbf{e}_k \rangle b_{ij}^{\mathbf{u}} b_{kj}^{\mathbf{u}} \right) dA \geq \lambda_1 \int_M \langle \mathbf{X}, \mathbf{u} \rangle^2 |\boldsymbol{\sigma}_r|^2 \, dA. \tag{12}$$

Theorem D follows by integrating both sides of (12) with respect to $\mathbf{X}$ over the unit sphere S^{n+p-1}, applying Proposition 3 and cancelling the appropriate constant factors.

Remark. Professor Weiner informs me that he can prove a result stronger than Theorem D for any nonzero parallel normal section.

Our final application of the main lemma is quite unlike the preceding ones.

THEOREM E. *Suppose that $\mathbf{Z}: M \to S^{n+p}$ is a minimal immersion of M into the unit sphere S^{n+p}. For any vector $\mathbf{C} \in S^{n+p}$ let $L_{\mathbf{C}}: S^{n+p} \sim \{\mathbf{C}\} \to \mathbf{R}^{n+p}$ be stereographic projection via the pole $\mathbf{C}$. Suppose that $\mathbf{C} \in S^{n+p} \sim \mathbf{Z}(M)$ and that the composite immersion $\mathbf{Y} = L_{\mathbf{C}} \circ \mathbf{Z}: M \to \mathbf{R}^{n+p}$ has the property that, in terms of the metric induced on M by $\mathbf{Y}$, $\int_M \mathbf{Y} \, dA = \mathbf{0}$. Then λ_1, the first eigenvalue of the Laplacian induced by $\mathbf{Y}$ on M, satisfies the inequality $\lambda_1 \leq n$. Moreover, $\lambda_1 = n$ if and only if $\mathbf{X}$ maps M into the totally geodesic equator of S^{n+p} perpendicular to $\mathbf{C}$.*

Proof. The main lemma says that $nA \geq \lambda_1 \int_M |\mathbf{Y}|^2 \, dA$. However, by Theorem 1 of our paper [5], the hypotheses of Theorem E imply that $\int_M |\mathbf{Y}|^2 \, dA \geq A$, with equality if and only if $\mathbf{X}$ maps M into the equator perpendicular to $\mathbf{C}$.

The desired result now follows easily.

BIBLIOGRAPHY

[1] Berger, M., Gauduchon, P. and Mazet, E., Le spectre d'une variété Riemannienne, Lecture Notes in Math. #194, Springer-Verlag (1971) Berlin.

[2] Blaschke, W., Vorlesungen über Differentialgeometrie, I. Springer (1924) Berlin.

[3] BLEECKER D. and WEINER, J., Extrinsic bounds on λ_1 of Δ on a compact manifold, Comment. Math. Helv. *51* (1976) 601–609.
[4] HSIUNG, C. C., Some integral formulas for closed hypersurfaces, Math. Scand. *2* (1954) 286–294.
[5] REILLY, R. C., Applications of stereographic projections to submanifolds in E^m and S^m, Proc. AMS *25* (1970) 119–123.
[6] ——, Variational properties of mean curvatures, Proc. 13th Biennial Seminar of the Canadian Math. Congress, vol. II (1973) 102–114.
[7] TAKAHASHI, T., Minimal immersions of Riemannian manifolds, J. Math. Soc. Japan *18* (1966) 380–385.

Mathematics Institute
University of Warwick
Coventry, Warwickshire CV4 7AL
England

Received December 24, 1976