Computation of Lojasiewicz Exponent of ...(x, y)

Computation of Lojasiewicz Exponent of $f(x, y)$[1]

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Let $f(x)$, $x \in \mathbf{R}^n$, $f(0)=0$, be a real analytic function defined near 0.

First, suppose $f(x)>0$ for $x \neq 0$. Then Lojasiewicz asserts that there exist $\varepsilon>0$, $\alpha>0$ such that

$$f(x) \geqslant \varepsilon |x|^{\alpha}, \qquad x \text{ near } 0.$$

Geometrically, this says that the graph of $y=f(x)$ lies above the bowl-like graph of $y=\varepsilon|x|^{\alpha}$.

For general $f(x)$, let V_f denote the variety $f(x)=0$ in $\mathbf{R}^n$, then Lojasiewicz ([2], p. 85; [3]) asserts that

$$|f(x)| \geqslant \varepsilon d(x, V_f)^{\alpha}, \qquad x \text{ near } 0,$$

where $d(\ ,\ )$ denotes the usual distance in $\mathbf{R}^n$.

This inequality, known as the Lojasiewicz inequality, is of fundamental importance in singularities theory.

PROBLEM. Determine the smallest value of α in the Lojasiewicz inequality.

For instance, to determine the smallest integer r such that in the Taylor expansion of f near 0, all terms of degree $> r$ can be omitted without changing the local topological type of f (i.e. the r-jet $j^{(r)}(f)$ is C^0-sufficient) amounts to determining α for $|\mathrm{Grad}\, f(x)|$ ([1], Theorem 0), where $\mathrm{Grad}\, f(x) \neq 0$ for $x \neq 0$.

Let $\mathbf{R}^2_+$ denote the upper half plane $\{(x, y): y \geqslant 0\}$; $\mathbf{R}^2_-$ the lower half plane $y \leqslant 0$.

All points are understood to be in a sufficiently small neighborhood of 0.

§1. The Results

Let $f(x, y)$ be a real analytic function of two real variables with $f(0, 0)=0$. We may assume, without loss of generality, that the initial form of the Taylor expansion of f is not divisible by y (this amounts to saying that the x-axis is not a tangent of $f(x, y)=0$ at 0). Then, by Puiseux's Theorem ([4], p. 98), f can be factored (near 0)

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[1]) The purpose of this paper is to give a complete solution of the Problem for functions $f(x, y)$ of two real variables. Our solution depends heavily on the use of Puiseux's Theorem.

into a product

$$f(x,y)=a\prod_{i=1}^{m}(x-z_i^*)\qquad a\neq 0 \tag{1.1}$$

where $m=O(f)$, the order of f, and each z_i^* is a fractional power series in y with order $O(z_i^*)\geqslant 1$.

EXAMPLE (1.2). $f(x,y)=x(x^2+y^3-y^4)$.
Roots are $z_1^*=0, z_2^*, z_3^*=\pm iy^{3/2}(1-\frac{1}{2}y+\cdots)$ with $O(z_1^*)=\infty, O(z_2^*)=O(z_3^*)=3/2$.

SOME DEFINITIONS. (i) Each z_i^* in (1.1) is called a root of f. A fractional power series is *real* if all coefficients are real. For a non-real z^*, $x=z^*$ has no locus in $\mathbf{R}^2_+$. This is because for $y>0$, all fractional powers of y are real numbers, and so z^* is not a real number. In $\mathbf{R}^2_-$, however, $x=z^*$ may, or may not, have locus. For the roots z_2^*, z_3^* in Example (1.2), there are loci in $\mathbf{R}^2_-$, the two arcs of a cusp. For the roots of $x^2+y^4=0$, there is no locus in either half plane (except the single point 0).

(ii) Let $z^*=\sum_{i=1}^{\infty}a_iy^{n_i}$ $a_i\neq 0$, $1\leqslant n_1<n_2<\cdots$ be a given non-real series, a_s its first non real coefficient. We define the *real springboard* of z^* to be

$$z^*(t)=\sum_{i\leqslant s-1}a_iy^{n_i}+ty^{n_s}$$

where t is a generic real number.

(1.3) Call n_s the complex order of z^*.

Now we put

$$e(f,z^*)=O(f(z^*(t),y))$$

and

$$\delta(f,z^*)=\begin{cases}\max\limits_i\{O(z^*(t)-z_i^*)\}, \text{ where } z_i^* \text{ runs through all real roots}\\ 1 \text{ if there is no real root.}\end{cases}$$

EXAMPLE (1.4). For $f(x,y)$ as in Example (1.2),

$$e(f,z_2^*)=O(t(t^2+1)y^{9/2}+\cdots)=9/2$$

for generic values of t. Notice that for some special value of t, such as $t=0$, $O(f(z_2^*(t),y))>9/2$.

We may call $e(f,z^*)$ the *climbing-exponent* of f along z^* and $\delta(f,z^*)$ the *distance-exponent* of f along z^*.

THEOREM. *The number*

$$L_+(f)=\operatorname*{Max}_j\left\{\frac{e(f, z_j^*)}{\delta(f, z_j^*)}, O(f)\right\}$$

where j runs through all indices for which z_j^* is a non real root, has the property that in any given sufficiently small compact neighborhood U of 0, *there exists a constant* $\varepsilon>0$ *such that*

$$(L_+) \quad |f(x, y)|\geqslant \varepsilon d((x, y), V_f)^{L_+(f)}, \qquad (x, y)\in U\cap R_+^2,$$

where $d(\ ,\ )$ denotes the usual distance in $\mathbf{R}^2$, *V_f the real curve $f(x, y)=0$. Moreover, $L_+(f)$ is the smallest number with this property.*

We put $L(f)\equiv \operatorname{Max}\{L_+(f), L_-(f)\}$, where $L_-(f)=L_+(f(x, -y))$, and call $L(f)$ the Lojasiewicz exponent of f.

COROLLARY 1. *The Lojasiewicz exponent $L(f)$ is the smallest number having the property that in any given sufficiently small compact neighborhood U of* 0 *in* $\mathbf{R}^2$, *there exists a constant* $\varepsilon>0$ *such that*

$$(L) \quad |f(x, y)|\geqslant \varepsilon d((x, y), V_f)^{L(f)}, \qquad (x, y)\in U.$$

COROLLARY 2. *If all roots of $f(x, y)$ (respectively $f(x, -y)$)* are real, then $L_+(f)=O(f)$ *(respectively $L_-(f)=O(f)$).*

COROLLARY 3. *If all roots of $f(x, y)$ are non-real, then* $L_+(f)=\operatorname{Max}_{1\leqslant i\leqslant m}\{e(f, z_i^*)\}$.

EXAMPLE. $f(x, y)=x^2+y^3$. Both roots of $f(x, y)$ are non-real, $L_+(f)=3$. Both roots of $f(x, -y)$ are real, $L_-(f)=2$. Hence $L(f)=3$.

COROLLARY 4. *$L(f)$, $L_+(f)$, $L_-(f)$ are rational numbers.*

§2. Proof of Theorem

Notations. For two real-valued functions $A(x_1,\dots, x_n)>0$ and $B(x_1,\dots, x_n)>0$ defined for $(x_1,\dots, x_n)$ in a domain D in $\mathbf{R}^n$ with $0\in\bar{D}-D$, we write

$$A\gtrsim B \tag{2.1}$$

if there exists a constant $k>0$ such that $kA\geqslant B$ for all $(x_1,\dots, x_n)$ in D near 0. If

$A \gtrsim B \gtrsim A$, then we write $A \sim B$; this is the case if, and only if A/B lies between two positive constants, x near 0.

For a fractional power series y^* in y, and for $d > 0$, $w > 0$, a *horn neighborhood* of y^* of degree d and width w is the point set

$$H_d(y^*; w) = \{(x, y): |x - \bar{y}^*| \leqslant w|y|^d\},$$

where $\bar{y}^*$ is y^* with all terms of degree $> d$ omitted. We often write $H_d(y^*)$ instead of $H_d(y^*; w)$. This is a horn-shaped set with vertex 0, containing the point set $x - y^* = 0$, except the origin, in its interior. The definition given here is slightly different from that in [1] in that this new horn neighborhood is the closure of the old one; in particular, the origin is contained in the new but not in the old.

$$\text{For } (x, y) \in H_d(y^*; w),\ |x - y^*| > w|y|^d. \tag{2.2}$$

For two fractional power series y^*, z^* let $H(y^*, z^*)$ denote the horn neighborhood $H_d(y^*; w)$ where $d = O(y^* - z^*)$, w a sufficiently small number. Call $H(y^*, z^*)$ the horn neighborhood of y^* *against* z^*.

$$H(y^*, z^*) \cap H(z^*, y^*) = \{0\}. \tag{2.3}$$

LEMMA (2.4). *Let z^* be a given fractional power series (in y). For a finite set of fractional power series $\{y_1^*, \ldots, y_s^*\}$ and $d > 0$, there is a finite subset $\Sigma = \Sigma_d(y_1^*, \ldots, y_s^*)$ of $\mathbf{R}$ such that for $t \notin \Sigma$,*

$$O((z^* + ty^d) - y_i^*) \leqslant d, \quad 1 \leqslant i \leqslant s. \tag{2.5}$$

The case $s = 1$ is quite obvious, the rest of the proof is by an easy induction on s.

Now consider all real springboards $z_i^*(t_i)$ of the non-real roots z_i^* of f. Let d_i denote the complex order (see (1.3)) of z_i^*. By a repeated application of Lemma (2.4), we can choose specified real values for t_i in $z^*(t_i)$ such that

$$O(z_i^*(t_i) - z^*) \leqslant d_i, \ O(z_i^*(t_i) - z_j^*(t_j)) \leqslant d_i \quad i \neq j, \tag{2.6}$$

where z^* is any real root.

EXAMPLE (2.7). $f(x, y) = (x - y^2)(x^2 + y^4)^2$, $z_1^* = y^2$, $z_2^* = z_3^* = iy^2$, $z_4^* = z_5^* = -iy^2$.

We may choose any values for t_i, $2 \leqslant i \leqslant 5$, provided that $t_i \neq 1$, and $t_i \neq t_j$ for $i \neq j$. Note that $t_2 = t_3$ or $t_4 = t_5$ are not allowed.

LEMMA (2.8). *Let $y^* = z^*(t)$ be the real springboard of a non-real fractional* *power series z^* with complex order $\leqslant d$. Then for $(x, y) \notin H_d(y^*; w)$, $(x, y) \in R^2_+$,*

$$|x-y^*| \sim |x-z^*|. \tag{2.9}$$

Proof. Let c be the first non real coefficient of z^*. First, suppose $d = O(y^* - z^*) =$ = complex order of z^*. Then

$$|x-z^*| \leqslant |x-y^*| + |y^*-z^*| \leqslant |x-y^*| + 2|c-t|\,|y|^d.$$

By (2.2), $|x-y^*| > w|y|^d$, hence

$$|x-z^*| \leqslant |x-y^*| + (2|c-t|/w)\,|x-y^*|.$$

Hence

$$|x-z^*| \lesssim |x-y^*|.$$

Now,

$$|x-y^*| \leqslant |x-z^*| + |z^*-y^*| \leqslant |x-z^*| + 2|c-t|\,|y|^d.$$

For $y>0$, y^d is real, while cy^d is non-real. Hence

$$|x-z^*| \geqslant \tfrac{1}{2}|c'|\,|y|^d,$$

where $c' = \operatorname{Im}(c)$, and so,

$$|x-y^*| \leqslant |x-z^*| + (4|c-t|/|c'|)\,|x-z^*|,$$
$$|x-y^*| \lesssim |x-z^*|, \quad \text{proving Lemma (2.8).}$$

Next, suppose $d < O(y^* - z^*)$. Then

$$\lim_{y \to 0} |y^* - z^*|\,|y|^{-d} = 0.$$

Again, (2.9) follows from the triangle inequalities

$$|x-y^*| \leqslant |x-z^*| + |z^*-y^*|$$

and

$$|x-z^*| \leqslant |x-y^*| + |y^*-z^*|.$$

From now on, $z_1^*, \ldots, z_m^*$ denote the m roots of f (with multiplicity), and let $y_1^*, \ldots, y_m^*$ denote the real roots and the springboards of non-real roots, satisfying (2.6).

That is, $y_i^*=z_i^*$ if z_i^* is real, and $y_i^*=z_i^*(t_i)$ if z_i^* is not real. All fractional power series are understood to be in y with order $\geqslant 1$. All points (x, y) are understood to be in $\mathbf{R}_+^2$.

Thus $H_d(y^*; w)\cap \mathbf{R}_+^2$ will be written simply as $H_d(y^*; w)$.

Let $\mathfrak{F}$ denote the family of horn neighborhoods of the y_i's against one another: $\mathfrak{F}=\{H(y_i^*, y_j^*): i\neq j\}$.

We divide a neighborhood of 0 (in $\mathbf{R}_2^+$) into regions of three types. *Type 1*: Regions which are the smallest members of $\mathfrak{F}$. *Type 2*: Those of type $H-\bigcup_\alpha H_\alpha$, H, $H_\alpha\in\mathfrak{F}$, where H_α runs through all members of $\mathfrak{F}$ contained in H. *Type 3*: The complements of the union of all members of $\mathfrak{F}$.

Inequality (L_+) will be established in regions of each type.

For a real fractional power series w^*, let V_{w^*} denote the point set $x-w^*=0$.

LEMMA (2.10). *Let y^* be a real fractional power series. Then*

$$d((x, y), V_{y^*})\sim |x-y^*|. \tag{2.11}$$

This is obvious.

LEMMA (2.12). *Let y^*, z^* be two real fractional power series. Then for $(x, y)\notin H(y^*, z^*)$,*

$$d((x, y), V_{y^*})\gtrsim d((x, y), V_{z^*}). \tag{2.13}$$

Consequently, for $(x, y)\notin H(y^*, z^*)\cup H(z^*, y^*)$,

$$d((x, y), V_{y^*})\sim d((x, y), V_{z^*}). \tag{2.14}$$

Proof. To show (2.13), let us first consider the special case $y^*=0$, whence V_{y^*} is the y-axis. Write $z^*=ay^d+\cdots$, $a\neq 0$. Now, by (2.11), $d((x, y), V_{y^*})\sim x$, $d((x, y), V_{z^*})\sim |x-(ay^d+\cdots)|$. For $(x, y)\notin H(y^*, z^*)=H_d(y^*)$, where $d=O(y^*-z^*)$, $|x|>$ $>w|y|^d$ by (2.2). Hence

$$|(x-(ay^d+\cdots))/x|=|1-(ay^d+\cdots)/x|\leqslant 1+2|a|\,|y|^d/w|y|^d=1+2|a|/w,$$

and (2.13) follows.

For the general case, we can perform a C^1-coordinate transformation

$$X=x-y^*,\qquad Y=y \tag{2.15}$$

(this transformation is C^1 since $O(y^*)\geqslant 1$). Then the general case is reduced to the above special case.

LEMMA (2.16). *Let y^*, z^* be two real fractional series. Then for $(x, y) \in H(y^*, z^*)$,*

$$d((x, y), V_{z^*}) \sim |y|^{\alpha}, \quad \alpha = O(y^* - z^*). \tag{2.17}$$

Proof. By (2.3), (2.2),

$$|x - z^*| > w|y|^{\alpha}.$$

Now

$$\begin{aligned} &H(y^*, z^*) = H_{\alpha}(y^*), \\ &|x - z^*| \leqslant |x - y^*| + |y^* - z^*| \\ &\qquad \leqslant w|y|^{\alpha} + |y^* - z^*| \sim |y|^{\alpha}. \end{aligned}$$

Hence (2.17) follows.

Type 1. Consider a smallest member H of $\mathfrak{F}$. Say $H = H(y_i^*, y_h^*)$.

First, suppose y_i^* is the real springboard of a non-real root z_i^*, $y_i^* = z_i^*(t_i)$.

Let d_i denote the complex order of z_i^* [(1.3)].

We claim that $H = H_{d_i}(y_i^*)$. Indeed, the complex conjugate $\bar{z}_i^*$ of z_i^* is another root of f, since f is real. Say $\bar{z}_i^* = z_k^*$. Now $O(z_k^* - z_i^*) = O(y_k^* - y_i^*) = d_i$, where $y_k^* = z_k^*(t_k)$, and so $H_{d_i}(y_i^*) = H(y_i^*, y_k^*)$ is a member of $\mathfrak{F}$. Since H is a smallest member, we must have $H \subset H_{d_i}(y_i^*)$. By (2.6), $H_{d_i}(y_i^*)$ is a smallest member of $\mathfrak{F}$; hence $H = H_{d_i}(y_i^*)$.

Now, for $(x, y) \in H$, $|x - z_i^*| \sim |y|^{d_i}$; moreover, for any $j \neq i$, $|x - y_j^*| \sim |y|^{\alpha_j}$, where $\alpha_j = O(y_j^* - y_i^*)$, by (2.17). If y_j^*, $j \neq i$, is the real springboard of a non-real root z_j^*, then we have $|x - z_j^*| \sim |x - y_j^*|$, by Lemma (2.8). Hence

$$|f(x, y)| = \prod_j |x - z_j^*| \sim |y|^{e(f, y^*_i)}, \tag{2.18}$$

since $e(f, y_i^*) = d_i + \sum_{j \neq i} \alpha_j$.

Now, V_f is defined by $\prod_j (x - y_j^*) = 0$ where y_j^* runs through all real roots. Hence

$$d((x, y), V_f) = \mathrm{Min}_j \{d((x, y), V_{y^*_j})\}.$$

For $(x, y) \in H = H_{d_i}(y_i^*)$,

$$d((x, y), V_{y^*_j}) \sim |x - y_j^*| \sim |y|^{\alpha_j}$$

where

$$\alpha_j = O(y_i^* - y_j^*) = O(y_i^* - z_j^*)$$

(See (2.6)). Hence

$$d((x, y), V_f) \sim |y|^{\delta(f, y_i^*)}. \tag{2.19}$$

By (2.18), (2.19), we have, for $(x, y) \in H_{d_i}(y_i^*)$

$$|f(x, y)| \sim d((x, y), V_f)^{\gamma_i} \tag{2.20}$$

where $\gamma_i = e(f, y_i^*)/\delta(f, y_i^*)$.

Next, suppose y_i^* is a real root. As y_i^* can be a multiple root, let μ denote its multiplicity. Hence

$$|f(x, y)| \sim |x - y_i^*|^{\mu} \prod_j |x - z_j^*|, \tag{2.21}$$

where z_j^* runs through all roots other than y_i^*. For $(x, y) \in H = H(y_i^*, y_h^*)$, we claim that

$$d((x, y), V_f) \sim d((x, y), V_{y^*_i}) \sim |x - y_i^*|. \tag{2.22}$$

Indeed, since H is a smallest member, H is contained in $H(y_i^*, y_s^*)$, and is therefore disjoint from $H(y_s^*, y_i^*)$, where y_s^* is any real root other than y_i^*. Then the first $\sim$ of (2.22) follows from (2.13), the second $\sim$ follows from (2.11). Moreover, for $(x, y) \in H$, and for $j \neq i$,

$$|x - z_j^*| \sim |x - y_j^*| \gtrsim |x - y_i^*|, \tag{2.23}$$

the first relation follows from Lemma (2.8), and the last relation follows from (2.13). Now, by (2.21), (2.23) and (2.22),

$$|f(x, y)| \gtrsim |x - y_i^*|^m \sim d((x, y), V_f)^m, \qquad m = O(f). \tag{2.24}$$

Remark. We may not replace $\gtrsim$ by $\sim$ in (2.24). See Example (3.6) in §3. However, for (x, y) in a sector $S_\eta = \{(x, y) : |y| \leqslant \eta |x|\}$ we do have

$$|f(x, y)| \sim d((x, y), V_f)^m. \tag{2.25}$$

In fact, since the initial form of f is not divisible by y, for $(x, y) \in S_\eta$, η sufficiently small,

$$|f(x, y)| \sim \varrho^m, \qquad \varrho = (x^2 + y^2)^{1/2} \sim |x|,$$

and

$$d((x, y), V_f) \sim |x| \sim \varrho.$$

Hence we have (2.25).

Type 2. The region is of the form $H - \bigcup_\alpha H_\alpha$. Collect all y_i^* for which $V_{y^*_i} \subset H$. By permutting the indices, if necessary, we may assume they are $y_1^*, \ldots, y_k^*$. Then

$V_{y^*_q} \not\subset H$, $q>k$. For $i \leqslant k$, $V_{y^*_i} \subset H$, then $H = H_{d_i}(y_i^*)$ for some d_i. Hence for $q>k$, $H(y_q^*, y_i^*) \cap H = \{0\}$.

First, suppose none of the y_i^*'s, $1 \leqslant i \leqslant k$, is a real root. Choose a fixed y_i^*, say y_1^*. For any j, $1<j\leqslant k$, $H(y_1^*, y_j^*)$ and $H(y_j^*, y_1^*)$ are disjoint proper subsets of H. Hence, by (2.11), (2.14),

$$|x-y_1^*| \sim |x-y_j^*|, \quad \text{for } (x, y) \in H - \bigcup_\alpha H_\alpha .$$

By (2.2),

$$|x-y_1^*| \gtrsim |y|^{\alpha_j}, \quad \alpha_j = O(y_1^* - y_j^*), \quad 1<j\leqslant k .$$

Moreover, by (2.6),

$$O(y_1^* - y_j^*) = O(y_1^* - z_j^*) \quad 1<j\leqslant k .$$

Hence, by Lemma (2.8),

$$\prod_{i=1}^{k} |x-z_i^*| \sim \prod_{i=1}^{k} |x-y_i^*| \gtrsim |y|^\alpha ,$$

where $\alpha = \sum_{i=1}^k O(y_1^* - z_i^*)$.

Now, for any $q>k$, $H(y_q^*, y_1^*) \cap H = \{0\}$. Hence for $(x, y) \in H$,

$$|x-z_q^*| \sim |x-y_q^*| \geqslant |y|^{\alpha_q}, \quad \alpha_q = O(y_1^* - y_q^*) = O(y_1^* - z_q^*) .$$

Thus, for $(x, y) \in H$,

$$|f(x, y)| \gtrsim |y|^{e(f, y_1^*)} . \tag{2.26}$$

We now show that

$$d((x, y), V_f) \sim |y|^{\delta(f, y_1^*)} \tag{2.27}$$

and then (L_+) follows.

In case f has no real root, $V_f = \{0\}$, $\delta(f, y_1^*) = 1$ and (2.27) is obvious.

Now let y_s^*, $s>k$, be any real root. We claim that $H \subset H(y_1^*, y_s^*)$. Indeed, H is a horn neighborhood of y_1^*, say of degree d_1. If $d_1 < O(y_1^* - y_s^*)$, then we would have $V_{y^*_s} \subset H$, a contradiction. Therefore $d_1 \geqslant O(y_1^* - y_s^*)$, $H \subset H(y_1^*, y_s^*)$. By Lemma (2.16),

$$d((x, y), V_{y_s^*}) \sim |y|^{\alpha_s}, \quad \alpha_s = O(y_1^* - y_s^*) . \tag{2.28}$$

Since y_s^* is any real root, (2.27) follows.

Next, suppose some y_i^*, $1 \leqslant i \leqslant k$, is a real root. Say $i=1$. By (2.11) and (2.14),

$$|x-y_1^*| \sim |x-y_j^*| \quad 2 \leqslant j \leqslant k.$$

For $q>k$,

$$|x-y_q^*| \gtrsim |x-y_1^*|$$

by (2.13). Therefore,

$$|f(x,y)| = \prod_{j=1}^{m} |x-z_j^*| \sim \prod_j |x-y_j^*| \gtrsim |x-y_1|^m \sim d((x,y), V_f)^m.$$

We have again established (L_+) in this case.

Type 3. First, suppose f has at least one real root. Say y_1^* is a real root. For $(x, y) \notin \bigcup_{i,j} H(y_i^*, y_j^*)$,

$$|x-y_i^*| \sim |x-y_j^*|, \quad \text{for all } i, j. \tag{2.29}$$

Moreover,

$$|f(x,y)| = \prod_i |x-z_i^*| \sim \prod_i |x-y_i^*| \sim |x-y_1^*|^m.$$

Now,

$$d((x,y), V_f) \sim \operatorname*{Min}_i \{|x-y_i^*|\} \sim |x-y_1^*|$$

by (2.29), where y_i^* runs through all real roots.
Therefore we have

$$|f(x,y)| \sim d((x,y), V_f)^m. \tag{2.30}$$

Finally, suppose there is no real root. We still have (2.29). Since $(x, y) \notin H(y_i^*, y_j^*)$ for all i, j,

$$|x-y_i^*| \gtrsim |y|^{\alpha_j} \quad \text{where} \quad \alpha_j = O(y_i^* - y_j^*), \quad j \neq i.$$

Hence

$$|f(x,y)| \gtrsim |y|^{e(y^*_i, f)}, \quad 1 \leqslant i \leqslant m.$$

Again, we have proved (L_+).

To complete the proof of the theorem, it remains to show that $L_+(f)$ is the smallest number having the property (L_+). This follows from (2.20) and (2.25).

Proof of the Corollaries. Corollary 1 follows immediately. Corollary 2 is obvious. Corollary 4 follows from Puiseux's Theorem; the exponents of the roots z_i^* are rational numbers (with a same denominator). Corollary 3 follows from the fact that $O(z_i^*)\geqslant 1$ for all i, and hence $e(f, z_i^*)\geqslant O(f)$.

§3. Illustrative Examples

For two arcs $\gamma: x=y^*$, $\beta: x=z^*$, call $d(\gamma, \beta)=O(y^*-z^*)$ the degree of contact of γ and β.

For a real arc $\gamma: x=y^*$, the Lojasiewicz exponent of f along γ, $l_f(\gamma)$, is defined by

$$|f(y^*, y)|\sim d((y^*, y), V_f)^{l_f(\gamma)}.$$

In particular, if $f(x, y)$ is positive definite, then

$$f(y^*, y)\sim |y|^{l_f(\gamma)}.$$

EXAMPLE (3.1). $f(x, y)=x^2+y^{10}$. Both roots are non-real. The real springboards of the roots are $\gamma_i: x=t_i y^5$, $i=1, 2$.

For any real arc β,

$$l_f(\beta)=\begin{cases}10 & \text{if } d(\gamma_i, \beta)\geqslant 5\\ 2d(\gamma_i, \beta) & \text{if } d(\gamma_i, \beta)\leqslant 5.\end{cases}$$

(3.2). As β varies so that $d(\gamma_i, \beta)$ increases, $l_f(\beta)$ increases.

The maximal value of $l_f(\beta)$ is 10 and is taken when $d(\gamma_i, \beta)\geqslant 5$.

Observe that $L(f)=10$ by Corollary 1.

A phenominon similar to (3.2) appears in the next example.

EXAMPLE (3.3). $f(x, y)=(x^2+y^{10})((x-y^3)^2+y^{40})$.

Consider the real springboard $\gamma_1: x=ty^5$, arising from the first factor, we have $l_f(\gamma_1)=16$. Let us perturb γ_1 to $\beta: x=ty^5+(sy^d+\text{terms of degree}>d)$, where $s\neq 0$, $|s|$ small. For $d=d(\gamma_1, \beta)$ varies in the range $1\leqslant d<\infty$,

$$l_f(\beta)=\begin{cases}16 & \text{if } d\geqslant 5\\ 2d+6 & \text{if } 5\geqslant d\geqslant 3\\ 4d & \text{if } 3\geqslant d\end{cases}$$

(3.4). As $d(\gamma_1, \beta)$ increases, $l_f(\beta)$ increases.

Observe that $l_f(\gamma_1)=16$ is a maximal value, which is reached when $d\geqslant 5$.

Now consider $\gamma_2: x=y^3+ty^{20}$, the real springboard of a root of the second factor.

For any $\beta : x = (y^3 + ty^{20}) + (sy^d + \cdots)$, we have

$$l_f(\beta) = \begin{cases} 46 & \text{if } \ d \geqslant 20 \\ 6+2\mathrm{d} & \text{if } \ 20 \geqslant d \geqslant 3 \\ 4d & \text{if } \ 3 \geqslant d \end{cases}$$

(3.5). Again, as $d = d(\gamma_2, \beta)$ increases, $l_f(\beta)$ increases.

Observe that $l_f(\gamma_2) = 46$ is a maximal value. Also, by Corollary 1, $L(f) = 46$.

When $f(x, y)$ has real roots, the way l_f varies near a real root is quite different from that near a non-real root as in the last two examples.

EXAMPLE (3.6). $f(x, y) = (x - y^2)(x^4 + y^{10})$.

Consider the value of $l_f(\beta)$, where

$$\beta : x = y^2 + (sy^d + \cdots), \qquad s \neq 0.$$

We have, along β,

$$|f(x, y)| = \begin{cases} |y|^{d+10} & d \geqslant 5/2 \\ |y|^{5d} & 5/2 \geqslant d, \end{cases}$$

$$d((x, y), V_f) \sim |y|^d.$$

Hence

$$l_f(\beta) = \begin{cases} 1 + 10/d & d \geqslant 5/2 \\ 5 & 5/2 \geqslant d. \end{cases}$$

Now observe that in contrast with (3.2), (3.4) and (3.5), $l_f(\beta)$ decreases as d increases.

The maximal value of l_f over arcs of type β is 5. Note that $O(f) = 5$. However, the maximal value of l_f near a real springboard of either root of the second factor is 12; and $L(f) = 12$.

In this example, $e(f, \gamma_i^*)/\delta(f, \gamma_i^*) = 12 > O(f) = 5$.

It is not true, however, that for general f, $e/\delta \geqslant O(f)$.

EXAMPLE (3.7). $f(x, y) = x(x - y^q)(x^2 + y^{2p})$, $p > q$. Then $e = p + q + 2p$, $\delta = p$, $O(f) = 4$, and $e/\delta < O(f)$.

To close this section, we give an example due to Lojasiewicz, which shows that for a polynomial f of degree n, one can have $L(f) > n$.

EXAMPLE (3.8). ([2], p. 85).

$$f(x, y) = x^{2n} + (x - y^n)^2 = (x - y^n + ix^n)(x - y^n - ix^n).$$

The roots are z_1^*, $z_2^* : x = y^n \pm iy^{n^2} + \cdots$. We have $e(f, z_i^*) = 2n^2 = L(f)$.

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Received September 25, 1973