Geodesics Satisfying General Boundary Conditions

Geodesics Satisfying General Boundary Conditions

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The existence of infinitely many geodesics joining orthogonal two submanifolds V_1 and V_2 of a Riemannian manifold M has been studied in Morse [4] and in Serre [5] under the assumptions V_1 compact and $V_1 \cap V_2 = \emptyset$. Existence of such geodesics is in that case clear. If however $V_1 \cap V_2 \neq \emptyset$ this is not in general the case (see §2). It is the purpose of this paper to examine that situation.

In §1 we shall study geodesics satisfying a general boundary condition. Special cases of this boundary condition are satisfied by $V_1 - V_2$-connecting geodesics, closed geodesics and isometry-invariant geodesics (see Grove [2], [3]). In §2 we concentrate on $V_1 - V_2$-connecting geodesics. A typical result in that section is that if V_1 and V_2 are compact and M is contractible then there exists $V_1 - V_2$-connecting geodesics.

The main tools in our approach are critical point theory on infinite dimensional manifolds and elementary homotopy theory.

§1. *N*-Geodesics

Throughout this paper M will be a connected complete Riemannian manifold and $N \subset M \times M$ a closed submanifold of $M \times M$. We shall say that *a geodesic* $\gamma : [0,1] \to M$ *is a N-geodesic if it satisfies the boundary condition*

$$(\gamma(0), \gamma(1)) \in N \quad \text{and} \quad (\dot{\gamma}(0), -\dot{\gamma}(1)) \text{ is normal to } N, \tag{1.1}$$

where $\dot{\gamma}(t)$ *denotes the velocity vector of* γ *at* t *and* $M \times M$ *is endowed with the product metric.*

EXAMPLES. (1) $\Delta(M)$-geodesics are closed geodesics on M.

(2) graph(A)-geodesics, $A : M \to M$ an isometry, are A-invariant geodesics (see [2] and [3]).

(3) $V_1 \times V_2$-geodesics are $V_1 - V_2$-connecting geodesics.

Let $L_1^2(I, M)$ denote the complete Riemannian Hilbertmanifold consisting of absolutely continuous curves $\sigma : I = [0, 1] \to M$ with $\dot{\sigma}$ square integrable (see Flaschel and Klingenberg [1]). From the propositions I.1.4 and I.1.5 of [2] it follows easily that

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[1]) This work was done under the program "Sonderforschungsbereich Theoretische Mathematik" (SFB 40) at the University of Bonn.

N-geodesics are in one to one correspondence with critical points for the energy integral $E: \Lambda_N(M) \to \mathbf{R}$, $\sigma \mapsto \frac{1}{2}\int_0^1 \|\dot{\sigma}\|^2$, *where* $\Lambda_N(M) = \{\sigma \in L_1^2(I, M) \mid (\sigma(0), \sigma(1)) \in N\}$ is a submanifold of $L_1^2(I, M)$. *Furthermore* $E: \Lambda_N(M) \to \mathbf{R}$ *satisfies condition* (C) *of Palais and Smale if projection on the first,* $P_1(N) \subset M$ *or the second factor* $P_2(N) \subset M$ *is compact* (Theorem I.2.4 of [2]). We shall therefore *assume that e.g.* $P_1(N) \subset M$ *is compact.* To make successful use of critical point theory for Hilbert-manifolds we *assume furthermore that* $N \cap \Delta$ *is a union of closed submanifolds,* – here Δ denotes the diagonal $\Delta(M)$ of M in $M \times M$.

With N as above we have,

LEMMA 1.2. *If there are no non-trivial N-geodesics on M, then the inclusion*

$$e: N \cap \Delta \to \Lambda_N(M), \qquad e(x, x)(t) = x \qquad \forall t \in I$$

is a homotopy equivalence.

Proof (*Sketch*). First we observe that each component of $N \cap \Delta$ is a compact non-degenerate critical submanifold of $\Lambda_N(M)$ of index 0. To see this we just note that the Hessian of $E: \Lambda_N(M) \to \mathbf{R}$ at a constant curve $\bar{p}: I \to M$, $\bar{p}(t) = p$ for all $t \in I$ is given by

$$H(E)_{\bar{p}}(X, Y) = \int_0^1 \langle X'(t), Y'(t) \rangle_p \, dt$$

for all $X, Y \in T_{\bar{p}}\Lambda_N(M)$. From this and $(X(0), X(1)) \in TN$ for $X \in T\Lambda_N(M)$ it easily follows that each component of $N \cap \Delta$ is a non-degenerate critical submanifold of index 0. We can now argue exactly as in the proofs of Corollary II.3 and Lemma II.4 of [2], i.e. by the generalized Morse Lemma and condition (C) prove that there exists an $\varepsilon > 0$ such that $N \cap \Delta$ is a strong deformation retract of $\Lambda_N(M)^\varepsilon$: $= \{\sigma \in \Lambda_N(M) \mid E(\sigma) < \varepsilon\}$. Assuming that there are no critical values > 0 (no non-trivial N-geodesics) we obtain from this, completenes of $\Lambda_N(M)$ and condition(C) that the inclusion $e: N \cap \Delta \to \Lambda_N(M)$ is a weak homotopy equivalence and hence a homotopy equivalence.

We are now ready to prove the main result of this section.

THEOREM 1.3. *If there are no non-trivial N-geodesics on M, then there is an exact sequence of homotopy groups*

$$\cdots - \pi_{*+1}(N) \xrightarrow{(P_1)_{*+1} - (P_2)_{*+1}} \pi_{*+1}(M) \to \pi_*(N \cap \Delta) \xrightarrow{i_*} \pi_*(N), \qquad * \geqq 0.$$

Exactness at $\pi_{*-1}(N\cap\Delta)$ implies that

$$\forall[k]\in\pi_*(M)\,\exists[g]\in\pi_*(N)\,\exists[h]\in\pi_*(N,N\cap\Delta)\quad\text{s.t.}$$
$$[k]=((P_1)_*([h])-(P_2)_*([h]))-((P_1)_*([g])-(P_2)_*([g]))\qquad *\geqq 1.$$

(for $*=1$ read multiplicative).

Proof. The inclusion $\Lambda_N(M)\to C_N^0(M):=\{f\in C^0(I,M)\mid(f(0),f(1))\in N\}$ is a homotopy equivalence (Theorem I.1.3 of [2]) so instead of $\Lambda_N(M)$ we consider $C_N^0(M)$. Consider now the commutative diagram,

$$\begin{tikzcd}
{} \arrow[r] & \pi_{*+1}(C_N^0M,\Omega M) \arrow[d, "P_{*+1}"', "\cong"] \arrow[dr, "\delta"] & & & \\
{} \arrow[r] & \pi_{*+1}(N) \arrow[r, "\partial"] \arrow[dr, "H\circ\partial"'] & \pi_*(\Omega M) \arrow[r, "j_*"] \arrow[d, "H"', "\cong"] & \pi_*(C_N^0M) \arrow[r, "P_*"] & \pi_*(N) \\
 & & \pi_{*+1}(M) \arrow[r] & \pi_*(N\cap\Delta) \arrow[u, "e_*", "\cong"'] \arrow[ur, "i_*"'] &
\end{tikzcd}$$

where the mid-sequence is the exact sequence for the fibration $P:C_N^0(M)\to N$, $P(f)=(f(0),f(1))$ with fiber the loopspace ΩM of M (Serre [5]), where δ is the boundary map in the exact sequence for the pair $(C_N^0(M),\Omega M)$ and H is the Hurewicz map. We shall compute the maps in cupic homotopi.

Let $\alpha\colon I^q\times I\to C_N^0(M)$ represent an element $[\alpha]$ in $\pi_{*+1}(C_N^0M,\Omega M)$. Evaluation of $\alpha_l\colon I^q\times\{0\}\to\Omega M$, $\hat{\alpha}_l$ represents $H\circ\delta[\alpha]=H\circ\partial(P_*[\alpha])$. From this we see that evaluation of $\alpha\colon I^{q+1}\to C_N^0M$, is a homotopy between $-P_1\circ P\circ\alpha+\hat{\alpha}_l+P_2\circ P\circ\alpha$ and the constant map, thus $H\circ\partial([\beta])=(P_1)_*([\beta])-(P_2)_*([\beta])$ for all $[\beta]=P_*[\alpha]\in\pi_{*+1}(N)$. Since we assume that there are no non-trivial N-geodesics $e_*\colon\pi_*(N\cap\Delta)\to\pi_*(C_N^0M)$ is an isomorphism by Lemma 1.2 and the lower sequence is the desired sequence.

We will now examine in detail what exactness at $\pi_*(N\cap\Delta)$ i.e. $e_*(\ker i_*)=im j_*$ means.

Let $[f]\in\pi_q(\Omega M)$ be represented by $f\colon I^q\to\Omega M$. Since $j_*([f])\in im e_*$ there is a homotopy

$$G_1\colon I^q\times I\to C_N^0(M)$$

with $G_1(\cdot,0)=j\circ f$ and $G_1(\cdot,1)\colon I^q\to e(N\cap\Delta)$. Identifying $e(N\cap\Delta)$ with $N\cap\Delta$ we get from $[G_1(\cdot,1)]\in e_*(\ker i_*)$ a homotopy

$$G_2\colon I^q\times I\to N$$

with $G_2(\cdot, 0)=G_1(\cdot, 1)$ and $G_2(\cdot, 1)=$base point. The homotopies $P\circ G_1: I^q\times I\to N$ and $G_2: I^q\times I\to N$ combines to an element $[g]\in\pi_{q+1}(N)$ and G_2 itself represents an element $[h]\in\pi_{q+1}(N, N\cap\Delta)$. Evaluation of $G_1, \hat{G}_1: I^q\times I\times I\to M$ give rise to a homotopy between $P_1\circ g+H(f)-P_2\circ g: I^{q+1}\to M$ and $P_1\circ h-P_2\circ h: I^{q+1}\to M$, note that $\hat{G}_1(\cdot, 0, \cdot)=H(f)$ i.e. $H[f]=(P_{1*+1}[h]-P_{2*+1}[h])-(P_{1*+1}[g]-P_{2*+1}[g])$. Q.E.D.

Remark. From theorem 1.3 we can derive the following theorems. *If M is compact there exist closed geodesics on M. If V is a compact submanifold of M and if there are no $V-V$-connecting geodesics then the inclusion $N\to M$ is a homotopy equivalence.* We can also obtain some partial results from [2] and [3] by this theorem.

§2. V_1-V_2-Connecting Geodesics.[2])

In this paragraph $N=V_1\times V_2$, where V_1 *and* V_2 *are closed connected submanifold of M, V_1 is compact and $V_1\cap V_2$ is a union of closed submanifolds of M* (may be of different dimensions). As mentioned in §1 is a $V_1\times V_2$-geodesic $\gamma: I\to M$ with $\gamma(0)\in V_1$, $\gamma(1)\in V_2$, $\dot{\gamma}(0)\perp V_1$ and $\dot{\gamma}(1)\perp V_2$.

We shall derive all our conclusions from the exact sequence of Theorem 1.3, which in the case $N=V_1\times V_2$ can be written as,

$$\to\pi_{*+1}(V_1\times V_2)\xrightarrow{(i_1)_{*+1}-(i_2)_{*+1}}\pi_{*+1}(M)\to\pi_*(V_1\cap V_2)\to\pi_*(V_1\times V_2) \tag{2.1}$$

We get immediately

COROLLARY 2.2. *Suppose that* $\dim(V_1\cap V_2)=0$ *and that* $\pi_1(M)=0$ *or* $V_1\cap V_2=\{pt\}$. *If there are no non-trivial V_1-V_2-connecting geodesics on M then all the homotopy groups of M are isomorphic to those of $V_1\times V_2$, – in fact $(i_1)_*-(i_2)_*$: $:\pi_*(V_1\times V_2)\to\pi_*(M)$ is an isomorphism.*

The following example illustrates this corollary.

EXAMPLE. Let V and W be Riemannian manifolds and let $M=V\times W$ be endowed with the product metric. For a fixed $(v, w)\in M$ put $V_1=V\times\{w\}$ and $V_2=$ $=\{v\}\times W$, then *there are no non-trivial V_1-V_2-connecting geodesics on M.* Other immediate consequences of (2.1) are

COROLLARY 2.3. *If $V_1\cap V_2$ is not connected and M is 1-connected then there exists non-trivial V_1-V_2-connecting geodesics.*

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[2]) Note that N-geodesics are in 1-1-correspondence with $N-\Delta(M)$-connecting geodesics in $M\times M$ with product metric (see e.g. L. N. Pattersen, On the index theorem, Amer. J. Math. *85* (1963), 271–297).

COROLLARY 2.4. *Suppose that M is a $K(\pi, 1)$ (e.g. M has negative curvature) and that there are no non-trivial $V_1 - V_2$-connecting geodesics on M. Then we have:*

(1) *If $\pi_1(V_1)=\pi_1(V_2)=0$ then $\pi_1(V_1 \cap V_2)=0$ and $\pi_1(M)$ is finite (impossible if M has neg. curvature).*

(2) *If $V_1 \cap V_2$ consists only of isolated points then $V_1 \times V_2$ is a $K(\pi', 1)$.*

COROLLARY 2.5. *Suppose that M is contractible. Then we have:*

(1) *If V_1 and V_2 are compact there exists non-trivial $V_1 - V_2$-connecting geodesics. ($V_1 = V_2 = \{pt\}$ not included).*

(2) *If V_2 is contractible and there are no non-trivial $V_1 - V_2$-connecting geodesics on M then $V_1 \subset V_2$.*

Proof. (1) From (2.1) we get that

$$i: V_1 \cap V_2 \to V_1 \times V_2, \quad i(p) = (p, p) \quad p \in V_1 \cap V_2$$

is a homotopy equivalence if there are no non-trivial $V_1 - V_2$-connecting geodesics. Since now $V_1 \times V_2$ is compact we get especially that $\dim(V_1 \cap V_2) = \dim(V_1 \times V_2)$ which is impossible except for the case $V_1 = V_2 =$ point.

(2) In the case V_2 contractible we obtain that

$$V_1 \cap V_2 \to V_1$$

is a homotopy equivalence if there are no non-trivial $V_1 - V_2$-connecting geodesics on M. By compactness $\dim(V_1 \cap V_2) = V_1$ which then implies that $V_1 \cap V_2 = V_1$ or equivalently $V_1 \subset V_2$.

Remark. $M = \mathbf{R}^3 \supset \mathbf{R}^2 = V_2 \supset V_1 = S^1$ gives an example of (2) in corollary 2.5. – It is clearly difficult to get more general results from (2.1). In concrete situations however where one knows more about homotopy groups of M, V_1, and V_2 (2.1) is useful in deciding whether there exists $V_1 - V_2$-connecting geodesics on M.

Let us finish with some remarks on the case where $M = \Sigma^n$ is a homotopy sphere.

COROLLARY 2.6. *Let $M = \Sigma^n$ be a homotopy sphere. Then*

(1) *If $V_1 \cap V_2$ consists only of isolated points there exists non-trivial $V_1 - V_2$-connecting geodesics on M.*

(2) *If there is a $q < n-1$ so that $\pi_q(V_1) \neq 0$ and $\pi_q(V_2) \neq 0$ then there exists non-trivial $V_1 - V_2$-connecting geodesics on M (more general $q < l-1$ if M is $(l-1)$-connected).*

Proof. Since $\max(\dim V_1, \dim V_2) < n$ we have that $\pi_q(V_1 \times V_2) \neq 0$ for some $q < n$. This together with corollary 2.2 proves (1). To prove (2) we see from (2.1) that $i_*: \pi_*(V_1 \cap V_2) \to \pi_*(V_1 \times V_2)$ is an isomorphism for $* < n-1$ if there are no non-trivial $V_1 - V_2$-connecting geodesics. If there is a $q<n-1$ with $\pi_q(V_1)\neq 0$ and $\pi_q(V_2)\neq 0$ this is clearly impossible.

To illustrate that no reasonable general existence results, besides those allready mentioned, can be expected, let us give one more example on non-existence of $V_1 - V_2$-connecting geodesics.

EXAMPLE. Let $M=S^3\subset \mathbf{R}^4$ with standard metric of constant curvature 1 and similar $V_2=S^2$ the equator of S^l. Then for any $V_1=S^1$ embedded in S^2 such that $V_1\cap(-V_1)=\emptyset$ there are no non-trivial $V_1 - V_2$-connecting geodesics. On the other hand we note that if $V_1=S^k$, $V_2=S^l$ and $V_i\backslash V_1\cap V_2\neq\emptyset$, $i=1,2$ then there exists non-trivial $V_1 - V_2$-connecting geodesics on $M=\Sigma^n$.

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Received April 24, 1973