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Connected Algebras and Universal Covering

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In his paper [4], Moore defined a *connected group* G to be one for which $G = [G, G]$ (thus, connected = perfect), and proved that every connected group has a unique simply connected covering group which is also its universal covering group. If we view $[G, G]$ as the verbal subgroup of G defined by the variety of abelian groups, then clearly the notion of connectedness can be generalised to other varieties, and in fact to other categories apart from groups. The basic notions involved are the invariants of Fröhlich's [1] (the Baer invariants) and the techniques used in that paper. For convenience, and to get away from groups, we shall work in the category $\mathscr{C}_\Lambda$ of associative algebras over some commutative ring Λ with identity. The algebras themselves need not have an identity.

For A, $B \in \mathscr{C}_\Lambda$, and $\phi : A \to \mathscr{M}(B)$ a homomorphism of A into the multiplication algebra $\mathscr{M}(B)$ of B, we assume that $\operatorname{Im}\phi$ is permutable on B. Let (B, A) denote the semi-direct product with multiplication defined by ϕ. Then $(B, A) \in \mathscr{C}_\Lambda$, and the exact sequence

$$\mathbf{E}(A, B; \phi) \equiv 0 \to B \to (B, A) \to A \to 0$$

realizes ϕ.

Suppose now that $B^2 = 0$. Let $\mathbf{E} \equiv 0 \to B \xrightarrow{\iota} E \xrightarrow{\pi} A \to 0$ be a singular extension, and suppose that we have a homomorphism $\psi : A \to E$ satisfying $\pi\psi = 1_A$. If $\mathbf{E}$ realizes ϕ, then $\mathbf{E}$ and $\mathbf{E}(A, B; \phi)$ are equivalent under the homomorphism $(B, A) \to E$ defined by $(b, a) \mapsto \iota(b) + \psi(a)$. If $\psi' : A \to E$ is another homomorphism which satisfies $\pi\psi' = 1_A$, then $\psi - \psi'$ induces a map $h : A \to B$ and h is a ϕ-crossed homomorphism (i.e., h is a homomorphism of Λ-modules, and $h(a_1 a_2) = \phi(a_1)\ h(a_2) + h(a_1)\ \phi(a_2)$). Clearly, such a ψ, if it exists, is unique if and only if $Z^1_\phi(A, B) = 0$.

PROPOSITION 1. *Suppose $B^2 = 0$. The following statements are equivalent.*

(i) *For every singular extension* $\mathbf{E}$ *of B by A which realizes ϕ, there exists a unique homomorphism $\psi : A \to E$ such that $\pi\psi = 1_A$.*

(ii) $H^2_\phi(A, B) = 0$ *and* $Z^1_\phi(A, B) = 0$.

(iii) $H^2_\phi(A, B) = 0$, $H^1_\phi(A, B) = 0$ *and* $H^0_\phi(A, B) = B$.

The proof is straightforward. For definitions, see [3].

Let $\mathscr{V}$ be a variety in $\mathscr{C}_\Lambda$. For $B \in \mathscr{V}$, suppose that $\phi : A \to \mathscr{M}(B)$ is $\mathscr{V}$-central (cf. [2]). Then $V(A) \subset \operatorname{Ker}\phi$, and so ϕ induces a homomorphism $\bar{\phi} : U(A) \to \mathscr{M}(B)$ and $\bar{\phi}$ is $\mathscr{V}$-central. For $h \in Z^1_\phi(A, B)$, consider the homomorphsim $\zeta_h : A \to (B, A)$

defined by $a \mapsto (h(a), a)$. Since $\mathbf{E}(A, B; \phi)$ realizes ϕ, it is $\mathscr{V}$-central, and so $V(\zeta_0) = V(\zeta_h)$. This implies that $V(A) \subset \operatorname{Ker} h$, and hence $Z^1_\phi(A, B) \cong Z^1_{\bar\phi}(U(A), B)$. This proves

PROPOSITION 2. *If $B \in \mathscr{V}$, and if $\phi : A \to \mathscr{M}(B)$ is $\mathscr{V}$-central, then*
(i) $Z^1_\phi(A, B) = Z^1_{\bar\phi}(U(A), B)$;
(ii) $A = V(A)$ *implies* $\phi = 0$ *and* $Z^1_0(A, B) = 0$.

An algebra A for which $A = V(A)$ will be called *$\mathscr{V}$-connected.* A $\mathscr{V}$-connected algebra A for which $H^2_0(A, B) = 0$ whenever $B \in \mathscr{V}$, $B^2 = 0$, will be called *$\mathscr{V}$-simply connected.* It follows then that

PROPOSITION 3. *If A is $\mathscr{V}$-connected and $B^2 = 0$, then a singular extension of B by A is $\mathscr{V}$-central if and only if $B \in \mathscr{V}$ and the extension is central.*

If E is $\mathscr{V}$-connected, we say that the surjective homomorphism $\pi : E \to A$ is a *$\mathscr{V}$-covering homomorphism* if $0 \to \operatorname{Ker}\pi \to E \to A \to 0$ is $\mathscr{V}$-central. Since U preserves surjections, this implies that A is itself $\mathscr{V}$-connected. Thus π is a $\mathscr{V}$-covering homomorphism if and only if $\operatorname{Ker}\pi \in \mathscr{V}$, and $\operatorname{Ker}\pi$ is central in E.

PROPOSITION 4. *If A is $\mathscr{V}$-connected, then A has a universal covering algebra.*

Proof. Let $\pi : \mathbf{E} \to A$ be a covering homomorphism of A, and let $\sigma : F \to E$ be a surjective homomorphism with F projective. Denote by K the kernel of the composite homomorphism $F \to E \to A$, and let $\mathbf{F} \equiv 0 \to K \to F \to A \to 0$. We obtain the commutative diagram

$$\begin{array}{ccccccccc}
0 \to & \dfrac{V(F) \cap K}{V_1(\mathbf{F})} & \to & \dfrac{V(F)}{V_1(\mathbf{F})} & \xrightarrow{\xi} & V(A) & \to 0 \\
 & \downarrow & & \downarrow \alpha & & \| & \\
0 \to & \dfrac{K}{V_1(\mathbf{F})} & \to & \dfrac{F}{V_1(\mathbf{F})} & \xrightarrow{\eta} & A & \to 0
\end{array} \qquad (1)$$

and, since $0 \to \operatorname{Ker}\pi \to E \to A \to 0$ is $\mathscr{V}$-central, the commutative diagram

$$\begin{array}{ccccccc}
0 \to & \dfrac{K}{V_1(\mathbf{F})} & \to & \dfrac{F}{V_1(\mathbf{F})} & \to & A & \to 0 \\
 & \downarrow & & \downarrow \gamma & & \| & \\
0 \to & \operatorname{Ker}\pi & \to & E & \to & A & \to 0
\end{array} \qquad (2)$$

where the vertical maps of (2) are induced by σ and are surjections.

The upper row of (1) is a $\mathscr{V}$-central extension of $A\,(= V(A))$, and so to show that it is a covering of A, we must show that $V(F)/V_1(\mathbf{F})$ is $\mathscr{V}$-connected. From the $\mathscr{V}$-centrality, and since F is projective, we can find a homomorphism $\beta: F/V_1(\mathbf{F}) \to V(F)/V_1(\mathbf{F})$ such that $\xi\beta=\eta$. This implies that $\eta=\eta\alpha\beta$ and $\xi\beta\pi=\xi$. Again from the $\mathscr{V}$-centrality of the rows of (1), we deduce that $V(\alpha\beta)=V(1_{F/V_1(F)})$ and that $V(\beta\alpha)= = V(1_{V(F)/V_1(F)})$. Thus $V(F)/V_1(\mathbf{F})=V\{V(F)/V_1(\mathbf{F})\}$, or in other words, $V(F)/V_1(\mathbf{F})$ is $\mathscr{V}$-connected.

Finally, to show universality, we prove that $V(\gamma): V(F)/V_1(\mathbf{F}) \to E$ is a $\mathscr{V}$-covering homomorphsim. Since $E=V(E)$ and V preserves surjections, then $V(\gamma)$ is a surjection. Actually, $V(\gamma)=\gamma\alpha$, and $\operatorname{Ker}\gamma\alpha \subset V(F)\cap K/V_1(\mathbf{F})$, and so $\operatorname{Ker}\gamma\alpha \in \mathscr{V}$ and

$$0 \to \operatorname{Ker}\gamma\alpha \to \frac{V(F)}{V_1(\mathbf{F})} \to E \to 0 \tag{3}$$

is a central extension (recall that the first row of (1) is a central extension). By Proposition 3, (3) is $\mathscr{V}$-central. This completes the proof.

The algebras $V(F)\cap K/V_1(\mathbf{F})$ and $V(F)/V_1(\mathbf{F})$ are the Baer-invariants of A, defined by Fröhlich in [1], and are denoted by $D_1U(A)$ and $D_0V(A)$ respectively. They do not depend on the choice of F, and in a natural sense, they generalise the idea of the Schur multiplier. At first sight, the covering homomorphism $\xi: D_0V(A) \to A$ appears to depend on π and on σ. That this is not the case was proved by Fröhlich in [1].

PROPOSITION 5. *The universal covering of a $\mathscr{V}$-connected algebra is unique.*

Proof. Let $\tau: X \to A$, $\tau': Y \to A$ be both universal coverings of A. Then there exist covering homomorphisms $\varrho: X \to Y$, $\varrho': Y \to X$ such that $\tau'\varrho=\tau$ and $\tau\varrho'=\tau'$. Hence $\tau'\varrho\varrho'=\tau$, and since $0 \to \operatorname{Ker}\tau' \to Y \to A \to 0$ is $\mathscr{V}$-central, we have $V(\varrho\varrho')=V(1_Y)$. But $Y=V(Y)$, and so $\varrho\varrho'=1_Y$. Similarly, we prove that $\varrho'\varrho=1_X$.

PROPOSITION 6. *If the $\mathscr{V}$-connected algebra A has a $\mathscr{V}$-simply connected covering algebra X, then X is the universal cover of A. Furthermore, every $\mathscr{V}$-covering of A has X as universal cover.*

Proof. Suppose $\tau: X \to A$ is a $\mathscr{V}$-cover of A, where X is $\mathscr{V}$-simply connected, and let $\tau': Y \to A$ be the universal cover of A (thus $Y=D_0V(A)$, by Proposition 4). By universality, there exists a $\mathscr{V}$-covering $\varrho': Y \to X$ such that $\tau\varrho'=\tau'$. From $\mathscr{V}$-simply connectedness of X, there exists a (unique) homomorphism $\varrho: X \to Y$ such that $\varrho'\varrho=1_X$. Then use Lemma 7.

Suppose that $E \to A$ is a $\mathscr{V}$-covering of A. Then this induces $\mathscr{V}$-covering homomorphisms $D_0V(A) \to E$ and $D_0V(E) \xrightarrow{\alpha} D_0V(A)$. By the first part, $D_0V(A)$ is $\mathscr{V}$-simply connected, and therefore $0 \to \operatorname{Ker}\alpha \to D_0V(A) \to 0$ is $\mathscr{V}$-central and splits. By Lemma 7, $D_0V(E) \cong D_0V(A)$.

LEMMA 7. *If* $0 \to Z \to Y \xrightarrow{\alpha} X \to 0$ *is* $\mathscr{V}$*-central and splits, and if* X *and* Y *are* $\mathscr{V}$*-connected, then* α *is an isomorphism and* $Z=0$.

Proof. Let $\beta: X \to Y$ be the spliiting homomorphism, i.e. $\alpha\beta = 1_X$. Then $\alpha = \alpha\beta\alpha$, and from $\mathscr{V}$-centrality we have $V(1_Y) = V(\beta\alpha)$. From the connectedness, this implies that $1_Y = \beta\alpha$.

PROPOSITION 8. *A* $\mathscr{V}$*-connected algebra* A *is* $\mathscr{V}$*-simply connected if and only if* $D_1 U(A) = 0$.

Proof. If A is $\mathscr{V}$-simply connected, then the $\mathscr{V}$-central extension $0 \to D_1 U(A) \to D_0 V(A) \to A \to 0$ splits. Since both A and $D_0 V(A)$ are $\mathscr{V}$-connected, then by Lemma 7 $D_1 U(A) = 0$.

Conversely, suppose that $D_1 U(A) = 0$. Let $0 \to \mathrm{Ker}\,\pi \to E \xrightarrow{\pi} A = 0$ be a singular $\mathscr{V}$-central extension. Then, as in the proof of Proposition 4, we obtain commutative diagrams (1) and (2), where ξ is an isomorphism. The map $\gamma\alpha\xi^{-1}: A \to E$ will then be the required splitting map.

In the group case, where the variety concerned is the variety of abelian groups, Moore has shown that the universal covering of every connected group is a simply connected group. This is no longer true in our generalisation. Later on we shall exhibit a counter-example. It is of interest however to ask under what conditions will a simply connected covering exist. In view of Proposition 6, this is equivalent to asking when the universal cover $D_0 V(A)$ of A is $\mathscr{V}$-simply connected.

Suppose then that $A = V(A)$, and let

$$\mathbf{E} \equiv 0 \to B \to E \to D_0 V(A) \to 0 \tag{4}$$

be a $\mathscr{V}$-central extension with $B^2 = 0$. This means (Proposition 3) that $\mathbf{E}$ is a central extension and that $B \in \mathscr{V}$. Pulling up via $D_1 U(A) \to D_0 V(A)$, we get the commutative diagram whose rows and columns are exact

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & B \arrow[r] \arrow[d, equal] & E^* \arrow[r] \arrow[d] & D_1 U(A) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & B \arrow[r] & E \arrow[r] \arrow[d] & D_0 V(A) \arrow[r] \arrow[d, "\xi"] & 0 \\
 & & A \arrow[r, equal] \arrow[d] & V(A) \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd} \tag{5}$$

Put $\mathbf{E}^* \equiv 0 \to B \to E^* \to D_1 U(A) \to 0$, and $\mathbf{\Omega} \equiv 0 \to E^* \to E \to A \to 0$. Since $\mathbf{E}$ is $\mathscr{V}$-central, then so is **E***. If **E** splits, then so does **E***, and in this case, since $D_1 U(A) \in \mathscr{V}$, we have $E^* \in \mathscr{V}$. Furthermore, since $D_1 U(A) \to D_0 V(A)$ is central, then $\mathbf{\Omega}$ is a central extension, and hence is $\mathscr{V}$-central. Thus **E** splits implies $\mathbf{\Omega}$ is $\mathscr{V}$-central.

Conversely, suppose that $\mathbf{\Omega}$ is $\mathscr{V}$-central. Then (cf. (1), (2)) we have a homomorphism $D_0 V(A) \to E$ whose composite with $E \to A$ is ξ. Since the right-hand column of (5) is $\mathscr{V}$-central, and since $D_0 V(A)$ is $\mathscr{V}$-connected, then the composite map $D_0 V(A) \to E \to D_0 V(A)$ is the identity map on $D_0 V(A)$. Hence **E** splits. We have therefore proved

PROPOSITION 9. *If A is $\mathscr{V}$-connected, then the $\mathscr{V}$-central extension* (4) *splits if and only if $\mathbf{\Omega}$ is $\mathscr{V}$-central.*

PROPOSITION 10. *If $\mathscr{V}$ contains the variety $\mathscr{A}$ of zero algebras, then the universal covering of any $\mathscr{V}$-connected algebra is $\mathscr{V}$-simply connected.*

Proof. In view of Proposition 9, we have to prove that, given any $\mathscr{V}$-central extension (4), the resulting sequence $\mathbf{\Omega}$ is $\mathscr{V}$-central. Since $D_0 V(A)$ is $\mathscr{V}$-connected, then $E = B + V(E)$. Now $E^*E + EE^* \subset B$, and since $\mathscr{V} \supset \mathscr{A}$, then $V(E) \subset E^2$. Therefore $E^*E = E^*V(E) \subset E^*E^2 \subset BE = 0$. Similarly, $EE^* = 0$. Hence $\mathbf{\Omega}$ is a central extension. Since $\mathscr{V} \supset \mathscr{A}$, then $E^* \in \mathscr{V}$, and so $\mathbf{\Omega}$ is $\mathscr{V}$-central.

COROLLARY 11. *The universal cover of any $\mathscr{A}$-connected algebra is $\mathscr{A}$-simply connected.*

We conclude by considering the variety $\mathscr{V}_J$. Let J denote an ideal of Λ ,our background ring. Then $\mathscr{V}_J$ denotes the variety consisting of those algebras A for which $JA = 0$. For this variety, we have (cf. [1]):

(i) $V(A) = JA$,
(ii) $U(A) = \Lambda/J \otimes_\Lambda A$,
(iii) $D_0 V(A) = J \otimes_\Lambda A$,
(iv) $D_1 U(A) = \mathrm{TOR}_1^\Lambda(\Lambda/J, A)$,
(v) $D_0 V(A) \to V(A)$ is given by $j \otimes a \to ja$.

PROPOSITION 12. *Every $\mathscr{V}_J$-connected algebra which is finitely generated as a Λ-module is $\mathscr{V}_J$-simply connected.*

Proof. Suppose that A is finitely generated as Λ-module, and that $A = JA$. Then there exists $\mu \in J$ such that $a = \mu a$ for all $a \in A$. Under the epimorphism $J \otimes A \to JA$, $\Sigma\lambda_i \otimes a_i = \mu \otimes \Sigma\lambda_i a_i$ is mapped onto $\Sigma\lambda_i a_i$. Therefore the kernel of this epimorphism is 0, i.e. $\mathrm{TOR}_1^\Lambda(\Lambda/J, A) = 0$. In view of (iv) and Proposition 8, A is $\mathscr{V}_J$-simply connected.

If we take Λ to be the ring Z of rational integers, and J to be the principal ideal (p) generated by p, $p \neq 0$, then the abelian group $A = Q/Z$ (where Q denotes the rationals) is divisible, and so, regarding A as a ring with trivial multiplication, then A is $\mathscr{V}_{(p)}$-

connected. The universal cover of A is $(p)\otimes Q/Z$, which is not $\mathscr{V}_{(p)}$-simply connected since $\operatorname{Ker}\{(p)\otimes(p)\otimes Q/Z \to (p)\otimes Q/Z\} \neq 0$.

Finally, we state the following proposition, whose proof is straightforward.

PROPOSITION 13. *Suppose that $\mathscr{V}_1$ and $\mathscr{V}_2$ are two varieties such that $\mathscr{V}_1 \supset \mathscr{V}_2$.*

(i) *Every $\mathscr{V}_1$-connected algebra is also $\mathscr{V}_2$-connected.*

(ii) *Every $\mathscr{V}_1$-simply connected algebra is also $\mathscr{V}_2$-simply connected.*

(iii) *If $\mathscr{V}_1 \cap \mathscr{A} = \mathscr{V}_2 \cap \mathscr{A}$, then every $\mathscr{V}_1$-covering homomorphism of a $\mathscr{V}_1$-connected algebra is also a $\mathscr{V}_2$-covering homomorphism.*

(iv) *If $\mathscr{V}_1 \cap \mathscr{A} = \mathscr{V}_2 \cap \mathscr{A}$, then the $\mathscr{V}_1$-universal covering of a $\mathscr{V}_1$-connected algebra is also its $\mathscr{V}_2$-universal covering.*

COROLLARY 14. *If $\mathscr{V} \supset \mathscr{A}$, and if $A = V(A)$, then $D_0 V(A) = F^2/(FK+KF)$ and $D_1 U(A) = (F^2 \cap K)/(FK+KF)$, where $0 \to K \to F \to A \to 0$ is a projective presentation of A.*

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