A Vanishing Theorem in Homological Algebra

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1. Introduction

In this paper we shall study properties of modules M over the mod2 Steenrod algebra $\mathscr{A}$ which are useful when using the Adams spectral sequence [1]. Our main result is a vanishing theorem for $\mathrm{Ext}_{\mathscr{A}}(M)$ in terms of $H(M; P_t^s)$, where P_t^s are certain elements of $\mathscr{A}$ whose square is zero so that they act as differentials on M.

THEOREM 1.1. *If M is an $\mathscr{A}$-module and $P_{t_0}^{s_0}$ is the lowest degree P_t^s with $s<t$ such that $H(M; P_t^s)\neq 0$, then* $\mathrm{Ext}^{s,t}(M, \mathbb{Z}_2)=0$ *for $ds>t+c$, where $d=\deg(P_{t_0}^{s_0})$ and $c/(d-1)$ is approximately t_0-2.*

All modules in this paper are understood to be positively graded.

Vanishing theorems of this type were first proved by Adams [2], the case of Theorem 1.1 when $d=3$. In 1969 Anderson and Mahowald independently proved the case $d=6$ – that the Adams spectral sequence of an $\mathscr{A}_1$-free module vanishes above slope $s/(t-s)=\frac{1}{5}$. This was related to Mahowald's computation of the image of J [8], for if $\bar{S}$ is the sphere spectrum with its zero degree homotopy group killed and P is infinite real projective space, there is a map $P\to\bar{S}$ [8], whose mapping cone has $\mathbb{Z}_2$-cohomology free over $\mathscr{A}_1$. In this paper we shall employ the techniques which Anderson used to prove this important case.

The effective use of P_t^s-homologies was initiated by Adams and Margolis [4]. We shall use our methods to obtain a proof of the main theorem of [4] which avoids detailed computations using the Steenrod algebra.

THEOREM 1.2. *If M is an $\mathscr{A}$-module such that $H(M; P_t^s)=0$ for all $s<t$, then M is a free $\mathscr{A}$-module.*

The main tool is an exact sequence relating $\mathrm{Ext}_A(\mathbb{Z}_2, \mathbb{Z}_2)$ and $\mathrm{Ext}_B(\mathbb{Z}_2, \mathbb{Z}_2)$, when A is obtained from B by addition of one generator. This is particularly useful if a certain element in $\mathrm{Ext}_A^1(\mathbb{Z}_2, \mathbb{Z}_2)$ can be shown to be nilpotent. We can accomplish this for certain subalgebras of the Steenrod algebra by using a variation on the bar resolution. The Steenrod algebra is then constructed one generator at a time, using the exact sequence to prove a vanishing theorem for the cohomology of each subalgebra.

We wish to thank the referee for many helpful suggestions, and particularly for his very nice proof of Proposition 3.4, which is the one which we give here.

2. The Bar Construction of a Hopf Algebra

If A is a graded algebra over k, and if M, N are two graded left A-modules, we can define bigraded groups $\mathrm{Ext}_A^{s,t}(M, N)$ as follows. Let $\mathbf{P} \to M$ be a projective resoluton of M as an A-module, and let $\mathrm{Hom}_A^s(P_n, N)$ be the A-homorphisms from P_n to N which decrease degree by s. Then $\mathrm{Ext}_A^{s,t}(M, N)$ is the t-dimensional homology of $\mathrm{Hom}_A^s(\mathbf{P}, N)$.

Let A, A' be two graded modules over k, and let M, N (resp. M', N') be two modules over A (resp. A'). Then $M \otimes M'$ and $N \otimes N'$ are modules over $A \otimes A'$. If $\mathbf{P} \to M$, $\mathbf{P}' \to M'$ are projective resolutions, so is $\mathbf{P} \otimes \mathbf{P}' \to M \otimes M'$. This gives us a pairing

$$\mathrm{Hom}_A^s(\mathbf{P}, N) \otimes \mathrm{Hom}_{A'}^{s'}(\mathbf{P}', N') \to \mathrm{Hom}_{A \otimes A'}^{s+s'}(\mathbf{P}' \otimes \mathbf{P}, N \otimes N'),$$

and consequently a pairing

$$\mathrm{Ext}_A^{s,t}(M, N) \otimes \mathrm{Ext}_{A'}^{s',t'}(M', N') \to \mathrm{Ext}_{A \otimes A'}^{s+s',t+t'}(M \otimes M', N \otimes N').$$

If A is a co-commutative connected Hopf algebra over k in the sense of [9], the diagonal $\Delta : A \to A \otimes A$ is a map of algebras, and thus makes every $A \otimes A$-module into an A-module. Since Ext is contravariant in the ring variable, Δ, together with the pairing above, induces a pairing

$$\mathrm{Ext}_A^{s,t}(M, N) \otimes \mathrm{Ext}_A^{s',t'}(M', N') \to \mathrm{Ext}_A^{s+s',t+t'}(M \otimes M', N \otimes N').$$

This pairing turns $\mathrm{Ext}_A^{**}(k, k)$ into an algebra, where A acts on k through the augmentation $\varepsilon : A \to k$. This product is the product used by Adams in [3].

Notice that if M is any A-module, M is naturally isomorphic to $k \otimes M$. Since $\varepsilon : A \to k$ is A-linear, so is $\varepsilon \otimes M : A \otimes M \to M$. Also, $\Delta : A \to A$ is A-linear, since it is an algebra map. Thus we can form an augmented simplicial A-module $\mathscr{H}_A(M)$ which in degree n is $A^{n+1} \otimes M$, with face and degeneracy maps given by $d_i = A^i \otimes \varepsilon \otimes A^{n-i} \otimes M$, $s_i = A^i \otimes \Delta \otimes A^{n-i} \otimes M$. The degeneracy $s_{-1} = \eta \otimes A^n \otimes M$ is a k-linear contracting homotopy, so we see that augmented simplicial A-module $\mathscr{H}_A(M) \to M$ is acyclic.

To show that the associated chain complex of $\mathscr{H}_A(M)$ is always a free resolution of M, it is necessary and sufficient to show that for any M, $A \otimes M$ is free as an A-module.

Let ${}_\varepsilon M$ be the A-module obtained from the k-module structure of M and the algebra map $\varepsilon : A \to k$. Then $A \otimes ({}_\varepsilon M)$ is certainly free, for A acts on it by the formula $a(a' \otimes m) = (aa') \otimes m$. We define two maps $\varphi : A \otimes {}_\varepsilon M \to A \otimes M$, $\gamma : A \otimes M \to A \otimes {}_\varepsilon M$. The map φ is the following composition:

$$A \otimes {}_\varepsilon M \xrightarrow{\Delta \otimes M} A \otimes A \otimes {}_\varepsilon M \xrightarrow{A \otimes \alpha} A \otimes M$$

where α is the action of A on M. Both maps are A-linear, so φ is A-linear. The map γ is the following composition:

$$A \otimes M \xrightarrow{\Delta\otimes M} A \otimes A \otimes M \xrightarrow{A\otimes\chi\otimes M} A \otimes A \otimes M \xrightarrow{A\otimes\alpha} A \otimes_{\varepsilon} M$$

where $\chi: A \to A$ is the canonical antiautomorphism. The following proposition follows from the formula $\mu(\chi\otimes A)\Delta = \eta\varepsilon = \mu(A\otimes\chi)\Delta$ and the usual associative laws. The proof is elementary and is left to the reader.

PROPOSITION 2.1. *φ and γ are inverse to one another. Thus γ is A-linear.*

Notice that (2.1) implies that each term of $\mathscr{H}_A(M)$ is free, so the associated chain complex is a free resolution of M. We call $\mathscr{H}_A(M)$ the Hopf bar construction of A for M. One can use (2.1) to show that $\mathscr{H}_A(M)$ is isomorphic to the usual bar construction, though we shall not show this as we shall not need it.

Let $M^* = \mathrm{Hom}_k(M, k)$. Since there is a natural isomorphism $\mathrm{Hom}_A(A\otimes_\varepsilon M, k) \cong M^*$, we have, via φ, a natural isomorphism $\tau: \mathrm{Hom}_A(A\otimes M, k) \to M^*$. Since τ is natural, if $\lambda: M \to N$ is A-linear, $\tau\, \mathrm{Hom}_A(A\otimes\lambda, k) = \lambda^*\tau$.

PROPOSITION 2.2. *Under the isomorphism τ above, $\varepsilon\otimes A\otimes M: A\otimes A\otimes M \to A\otimes M$ has (as its dual $(\chi\otimes M^*)\alpha^*: M^* \to A^*\otimes M^*$, where $\alpha: A\otimes M \to M$ is the action of A on M. Also, $\Delta\otimes M: A\otimes M \to A\otimes A\otimes M$ has as its dual $\eta^*\otimes M^*: A^*\otimes M^* \to M^*$. Thus $\mathrm{Hom}_A(\mathscr{H}_A(M), k)$ is isomorphic to a cosimplicial k-module which in codegree n is $(A^*)^n\otimes M^*$, and with coface and cogeneracy maps given by:*

$$\begin{aligned}
d^i &= (A^*)^{i-1} \otimes \varepsilon^* \otimes (A^*)^{n-i+1} \otimes M^* && i > 0\\
s^i &= (A^*)^{i-1} \otimes \Delta^* \otimes (A^*)^{n-i-1} && i > 0\\
s^0 &= \eta^* \otimes (A^*)^n \otimes M^*\\
d^0 &= ((\chi(\Delta^{n+1})^*) \otimes (A^*)^n \otimes M^*)\, T^{n+1}((\mu^*)^n \otimes \alpha^*)
\end{aligned}$$

where $\Delta^{n+1}: A \to A^{n+1}$ is the iterated diagonal, and T^{n+1} is the signed interchange

$$T^{n+1}(a_1 \otimes a_1' \otimes \cdots \otimes a_{n+1} \otimes m) = \pm\, a_1 \otimes a_2 \otimes \cdots \otimes a_{n+1} \otimes a_1' \otimes \cdots \otimes a_n' \otimes m\,.$$

Given any simplicial abelian group G, Moore has shown how to define an associated chain complex $\bar{G}$ with homology the same as that of G. Let $\bar{G}_n \subset G_n$ be $\mathrm{Ker}(d_1) \cap \ldots \cap \mathrm{Ker}(d_n)$, and let $d: \bar{G}_n \to \bar{G}_{n-1}$ be d_0. Then $d^2 = 0$ on $\bar{G}$, and the obvious map $H_n(\bar{G}) \to H_n(G)$ is an isomorphism. Similarly, if G is a cosimplicial abelian group, let $\bar{G}_n = \bar{G}_n/I_m(d^1) + \cdots + I_m(d^n)$, and let $d: \bar{G}_n \to \bar{G}\eta_{+1}$ be d^0. Then $H^n(G) \to H^n(\bar{G})$ will be an isomorphism.

Let $G_n = \mathrm{Hom}_A(\mathscr{H}_A(k)_n, k)$. Then every element of G_n can be represented in the form $(\alpha_1|\ldots|\alpha_n)$ where $\alpha_i \in A^*$ all i. In $\bar{G}_n$ we set equal to 0 all $(\alpha_1|\ldots|\alpha_n)$ with any $\alpha_i \in \mathrm{Im}(\varepsilon^*)$, which is to say $\deg(\alpha_i)=0$. Notice that

$$d(\alpha_1|\ldots|\alpha_n) = \sum \pm (\chi(\alpha'_{1\nu_1}\ldots\alpha'_{n\nu_n})\,|\alpha''_{1\nu_1}|\ldots|\alpha''_{n\nu_n}) \quad \text{where} \quad \mu^*(\alpha_i) = \sum \alpha'_{i\nu_i} \otimes \alpha''_{i\nu_i}.$$

Since the degreewise tensor product of $\mathscr{H}_A(M)$ with $\mathscr{H}_A(M')$ is $\mathscr{H}_{A\otimes A}(M\otimes M')$, we can use the Alexander-Whitney map to define a chain homotopy equivalence $\mathscr{H}_{A\otimes A}(M\otimes M') \to \mathscr{H}_A(M)\otimes\mathscr{H}_A(M')$. Recall that the Alexander-Whitney map is defined, for x, y of degree n, by $AW(x\otimes y) = \sum (d_i^{n-i+1}x)\otimes(d_0^i y)$. By direct calculation, it is easy to verify that the composite with the diagonal $\mathscr{H}_A(k)\to\mathscr{H}_{A\otimes A}(k)$ gives us a map in degree n which is the product of the maps $A^{n+1}\to A^{i+1}\otimes A^{n-i+1}$ given by $A^i\otimes\Delta\otimes A^{n-i}$ for $0\leqslant i\leqslant n$. If we apply (2.1), we obtain the following.

PROPOSITION 2.3. *The product in* $\mathrm{Ext}_A(k,k)$, *relative to the basis of cochains given before, is given by*

$$(\alpha_1|\ldots|\alpha_i)(\alpha_{i+1}|\ldots|\alpha_n) = \sum (\alpha_1|\ldots|\alpha_{i-1}|\alpha_i\chi(\alpha'_{i+1,\nu_{i+1}}\ldots\alpha'_{n,\nu_n})\,|\alpha''_{i+1,\nu_{i+1}}|\ldots|\alpha''_{n,\nu_n})$$

where

$$\mu^*(\alpha_j) = \sum \alpha'_{j,\nu_j}\otimes\alpha''_{j,\nu_j}.$$

COROLLARY 2.4. *If* $\alpha_{i+1},\ldots,\alpha_n$ *are primitive in* A^*, *then for any* $\alpha_1,\ldots,\alpha_i$, $(\alpha_1|\ldots|\alpha_i)\,(\alpha_{i+1}|\ldots|\alpha_n) = (\alpha_1|\ldots|\alpha_n)$. *In particular, if* α *is primitive,* $(\alpha)^n = (\alpha|\ldots|\alpha)$.

DEFINITION 2.5. [4] Let P_t^s be the Milnor basis element $Sq(0,\ldots,0,2^s,0,\ldots)$ with the 2^s in the tth position.

PROPOSITION 2.6. [5] *The sub-Hopf-algebras of* $\mathscr{A}$ *correspond bijectively to sequences of nonnegative integers* $p_1,\ldots p_i$ *(possibly equal to* ∞*) satisfying* $p_i \geqslant \min(p_j, p_{i-j}-j)$ *wherever* $i>j$. *The algebra* $A(p_1,\ldots)$ *corresponding to such a sequence has as* $\mathbb{Z}_2$*-basis all* $Sq(r_1,\ldots)$ *with* $r_i<2P_i$.

We will find it convenient to use $\zeta_i=\chi(\xi_i)$ as the generators of $\mathscr{A}^*$. $\Delta(\zeta_i) = \sum_{j=0}^{i}\zeta_j\otimes\zeta_{i-j}^{2^j}$. The dual of $A(p_1,\ldots)$ is $\mathbb{Z}_2[\zeta_1,\ldots]/\zeta_i^{p_i}=0$.

THEOREM 2.7. *Suppose m and n are nonnegative integers and* $A=A(p_1,\ldots)$ *is a sub-Hopf-algebra with* $p_i\leqslant n$ *for all* $i\leqslant m$, $p_{m+1}=m+n+2$, *and* $p_{2m+2}\geqslant n+1$. *Then* $[\zeta_{m+1}^{2^{m+n+1}}]$ *is a nilpotent element of* $\mathrm{Ext}_A(\mathbb{Z}_2,\mathbb{Z}_2)$.

Proof. $\Delta(\zeta_{m+1}^{2^{m+n+1}}) = \sum_{j=0}^{m+1}\zeta_j^{2^{m+n+1}}\otimes\zeta_{m+1-j}^{2^{m+n+1+j}} = \zeta_{m+1}^{2^{m+n+1}}\otimes 1 + 1\otimes\zeta_{m+1}^{2^{m+n+1}}$ since $\zeta_j^{2^n}=0$ for $j\leqslant m$. Thus in the dual of the Hopf bar resolution $d^*([\zeta_{m+1}^{2^{m+n+1}}])=0$, and hence $[\zeta_{m+1}^{2^{m+n+1}}]\in \mathrm{Ext}_A^1(\mathbb{Z}_2, \mathbb{Z}_2)$. To show its nilpotence, we shall write $[\zeta_{m+1}^{2^{m+n+1}}]^{2^{m+2}} = [\zeta_{m+1}^{2^{m+n+1}}|\dots|\zeta_{m+1}^{2^{m+n+1}}]$ as a boundary. Let T be the element of $\overline{A}^{*2^{m+2}-1}$ in which $\zeta_{2m+2}^{2^n}$ occurs in the first $2\eta^{+1}$ positions and $\zeta_{m+1}^{2^{m+n+1}}$ occurs in the rest. Since $\Delta(\zeta_{2m+2}^{2^n}) = \zeta_{2m+2}^{2^n}\otimes 1 + 1\otimes\zeta_{2m+2}^{2^n} + \zeta_{m+1}^{2^n}\otimes\zeta_{m+1}^{2^{m+n+1}}$, $d^*(T)$ is the sum of all terms having $\chi(\zeta_{m+1}^{i2^n})$ $(i\geq 1)$ in the first position, $\zeta_{2m+2}^{2^n}$ in some $2^{m+1}-i$ of the next 2^{m+1} positions, and $\zeta_{m+1}^{2^{m+n+1}}$ in all other positions. Let S be the sum of all $\binom{2^{m+2}-1}{2^{m+1}}$ terms having $\zeta_{2m+2}^{2^n}$ in 2^{m+1} positions and $\zeta_{m+1}^{2^{m+n+1}}$ in $2^{m+1}-1$ positions of an element of $\overline{A}^{*2^{m+2}-1}$. Then $d^*(S) = [\zeta_{m+1}^{2^{m+n+1}}|\dots|\zeta_{m+1}^{2^{m+n+1}}]$. To see this note that $[\chi(\zeta_{m+1}^{2^{m+n+1}})|\dots|\zeta_{m+1}^{2^{m+n+1}}]$ occurs an odd number $(\binom{2^{m+2}-1}{2^{m+1}})$ of times in the sum, and

$$\begin{aligned}\chi(\zeta_{m+1}^{2^{m+n+1}}) &= \chi(\zeta_{m+1})^{2^{m+n+1}} = (\zeta_{m+1} + p(\zeta_1,\dots,\zeta_m))^{2^{m+n+1}}\\ &= \zeta_{m+1}^{2^{m+n+1}} + p(\zeta_1^{2^{m+n+1}},\dots,\zeta_m^{2^{m+n+1}}) = \zeta_{m+1}^{2^{m+n+1}}.\end{aligned}$$

Any term having $\zeta_{m+1}^{2^{m+n+1}}$ in some fixed $2^{m+1}-1+i\,(0<i<2^{m+1})$ of the positions occurs in the image under d^* of each term having its $2^{m+1}-1\zeta_{m+1}^{2^{m+n+1}}$'s in some of these positions; there are $\binom{2^{m+1}-1+i}{2^{m+1}-1}$ such terms, and this is an even number.

This also shows $[\xi_{m+n}^{2^{m+n+1}}]$ nilpotent, since $\xi_{m+1}^{2^{m+n+1}} = \chi(\zeta_{m+1}^{2^{m+n+1}}) = \zeta_{m+1}^{2^{m+n+1}}$.

3. Techniques that Will be Used to Prove the main Theorems

DEFINITION 3.1. If A is an algebra over k, M an A-module, and $z\in\mathrm{Ext}_A(k,k)$, we say *z acts nilpotently* on $\mathrm{Ext}_A(M,k)$ if every element of $\mathrm{Ext}_A(M,k)$ is annihilated by some power of z.

Our main technique will be to build up the algebras by using

THEOREM 3.2. *Suppose B is a subalgebra of A such that A is a free B-module on two generators,* 1, x, *of degrees* 0, n. *Then there is a long exact sequence for any A-module M*

$$\to \mathrm{Ext}_A^{s,t+n}(M,k) \to \mathrm{Ext}_B^{s,t+n}(M,k) \to \mathrm{Ext}_A^{s,t}(M,k) \xrightarrow{\cdot[\hat{x}]} \mathrm{Ext}_A^{s+1,t+n}(M,k) \to ,$$

where $[\hat{x}]\in\mathrm{Ext}_A^{1,n}(k,k)$ corresponds to the extension

$$0\to k\to \mathrm{Hom}_B(A,k)\to k\to 0.$$

Moreover, if $[\hat{x}]$ acts nilpotently on $\mathrm{Ext}_A(M,k)$, *then*

(i) *If M is a free B-module, then M is a free A-module.*

(ii) *If* $\mathrm{Ext}_B^{s,t}(M,k)=0$ *for $ds>t+c$ with $n\leq d$, then* $\mathrm{Ext}_A^{s,t}(M,k)=0$ *for $ds>t+$ $+c+n$.*

Proof. The exact sequence follows from the exact sequence

$$\to \operatorname{Ext}_A^{s,t+n}(M,k) \to \operatorname{Ext}_A^{s,t+n}(M, \operatorname{Hom}_B(A,k)) \to \operatorname{Ext}_A^{s,t}(M,k)$$
$$\to \operatorname{Ext}_A^{s+1,t+n}(M,k) \to$$

and the change-of-rings isomorphism [6; Proposition 4.1.4] $\operatorname{Ext}_A(M, \operatorname{Hom}_B(A,k)) \approx \approx \operatorname{Ext}_B(M,k)$. The differential is due to [3; Theorem 2.6.1.] If $y \in \operatorname{Ext}_A^{s,t}(M,k)$, then for some $m \geqslant 0$ $[\hat{x}]^m y$ is a nonzero element of $\ker(\operatorname{Ext}_A^{s+m,t+nm}(M,k) \xrightarrow{\cdot[\hat{x}]} \operatorname{Ext}_A^{s+m+1,t+(m+1)n}(M,k))$ and hence pulls back to an element of $\operatorname{Ext}_B^{s+m,t+n(m+1)}(M,k)$.

(i) If M is B-free, $\operatorname{Ext}_B^{s+m,t+nm}(M,k)=0$ for $s+m>0$. Thus y cannot exist unless $s=0$. Thus M is A-free.

(ii) If the element induced by y is to exist, we must have $t+n(m+1)+c \geqslant d(s+m)$. Therefore $t+c+n \geqslant ds+m(d-n) \geqslant ds$. ∎

Theorem 3.2 will be applied both to the nilpotent elements found in Theorem 2.7 and to elements which will be seen to act nilpotently due to the following result.

PROPOSITION 3.3. *If* $\operatorname{Ext}_A^{s,t}(M,k)=0$ *for* $ds>t+c$ *and* $x \in \operatorname{Ext}_A^{1,n}(k,k)$ *with* $n<d$, *then* x *acts nilpotently on* $\operatorname{Ext}_A(M,k)$.

Proof. If $y \in \operatorname{Ext}_A^{s,t}(M,k)$ and $m>(t-ds+c)/(d-n)$, then $x^m y=0$. ∎

The hypothesis of Proposition 3.3 will often be verified using Proposition 3.4 below, which has as corollary that vanishing theorems of the type which we are seeking hold for modules over exterior algebras on generators of distinct degrees.

PROPOSITION 3.4. *Assume that* A *is free as a left module over the subalgebra* B, *with a* B-*base consisting of* 1 *and other elements* a_α *of degree at least* d. *Suppose also that* M *is a* B-*free module over* A *and that* $M_t=0$ *for* $t<c$. *Then* $\operatorname{Ext}_A^{s,t}(M,k)=0$ *for* $t<ds+c$.

Proof. By induction over s. Form the exact sequence

$$0 \to K \to A \otimes_B M \to M \to 0.$$

Let $\{m_\mu\}$ be a B-base for M; then $A \otimes_B M$ has an A-base $\{1 \otimes m_\mu\}$ and a B-base $\{1 \otimes m_\mu, a_\alpha \otimes m_\mu\}$. So K has a B-base $\{a_\alpha \otimes m_\mu - 1 \otimes a_\alpha m_\mu\}$. Thus $K_t=0$ for $t<d+c$. The inductive hypothesis shows that $\operatorname{Ext}_A^{s-1,t}(V,k)=0$ for $t<d(s-1)+d+c$. Since $\operatorname{Ext}_A^{s-1,t}(K,k)$ maps onto $\operatorname{Ext}_A^{s,t}(M,k)$, the latter is zero for $t<ds+c$. ∎

We shall apply Proposition 3.4 in cases when B is a sub-Hopf-Algebra of A, so that the first hypothesis is fulfilled by [9; Theorem 4.4].

If E is an exterior algebra on generators x_i of distinct degree, and M is an E-module, then the x_i act as differentials on M.

COROLLARY 3.5. *If E and M are as above and*

$$d = \min\{\deg x_i : H(M; x_i) \neq 0\},$$

then $\mathrm{Ext}_E^{s,t}(M, k) = 0$ *for* $ds > t$.

Proof. By [4; Theorem 2.1] both M and E are free over $E[\{x_i : \deg x_i < d\}]$, so the corollary follows from Proposition 3.4. ∎

We shall prove vanishing theorems over finite sub-Hopf-algebras and then conclude vanishing theorems over the entire Steenrod algebra by applying

PROPOSITION 3.6. *Suppose A is a sub-Hopf-algebra of $\mathscr{A}$ and $\overline{\mathscr{A}/\!/\mathscr{A}\bar{A}}$ begins in degree n. Suppose $x_1, \ldots, x_m$ are elements of A such that whenever $H(M; x_i) = 0$, $i = 1, \ldots, m$, $\mathrm{Ext}_A^{st}(M, \mathbb{Z}_2) = 0$ for $ds > t + c$ with $d \leqslant n$. If M is an $\mathscr{A}$-module such that $H(M; x_i) = 0$, $i = 1, \ldots, m$, then $\mathrm{Ext}_{\mathscr{A}}^{st}(M, \mathbb{Z}_2) = 0$ for $ds > t + c$.*

Proof. Let $W = \mathscr{A}/\!/\mathscr{A}\bar{A}$. The exact sequences

$$0 \to \bar{W} \otimes \bar{W}^{i-1} \otimes M \to W \otimes \bar{W}^{i-1} \otimes M \xrightarrow{f} \bar{W}^{i-1} \otimes M \to 0$$

(diagonal $\mathscr{A}$-action, $f(w \otimes p) = \varepsilon(w)p$), yield exact sequences

$$\begin{aligned} \to \mathrm{Ext}_{\mathscr{A}}^{s-i,t}(\bar{W}^i \otimes M, \mathbb{Z}_2) &\to \mathrm{Ext}_{\mathscr{A}}^{s-i+1,t}(\bar{W}^{i-1} \otimes M, \mathbb{Z}_2) \\ &\to \mathrm{Ext}_A^{s-i+1,t}(\bar{W}^{i-1} \otimes M, \mathbb{Z}_2) \to , \end{aligned}$$

where the Ext_A-group has been obtained because $\mathrm{Ext}_{\mathscr{A}}(w \otimes N, \mathbb{Z}_2) \approx \mathrm{Ext}_A(N, \mathbb{Z}_2)$. Thus an element of $\mathrm{Ext}_{\mathscr{A}}^{s,t}(M, \mathbb{Z}_2)$ must come from an element of $\mathrm{Ext}_A^{s-i,t}(\bar{W}^i \otimes M, \mathbb{Z}_2)$ for some i. Since M is x_i-acyclic, the same is true of $\bar{W}^i \otimes M$; this is easily seen even if x_i is not primitive. Since $\bar{W}^i \otimes N$ begins in degree *in* $\mathrm{Ext}_A^{s-i,t}(\bar{W}^i \otimes M, \mathbb{Z}_2) = 0$ for $d(s-i) > t - in + c$ and hence makes no contribution to $\mathrm{Ext}_{\mathscr{A}}^{s,t}(M, \mathbb{Z}_2)$ when $ds > t + c$. ∎

4. Freeness and Vanishing Theorems

In this section we shall prove the freeness theorem 1.2 and a vanishing theorem slightly weaker than Theorem 1.1. The proofs are done similarly, proving them first over an exterior algebra, and then building up the algebra one generator at a time, in an order such that new generators act nilpotently, until we have proved it over a sufficiently large finite subalgebra.

Let $A(O^i, p_{i+1}, p_{i+2}, \ldots)$ denote the algebra corresponding by Proposition 2.6 to a sequence beginning with i zeros. Let $B_{i,j,n} = A(O^i, j, n-i, n-i-1, \ldots, 1)$ if $i + j \leqslant n + 1$.

LEMMA 4.1. *Suppose M is an $\mathscr{A}$-module and suppose we are given integers i, j, n*

such that $i+j\leqslant n$. *Also, suppose that if* $j<i+1$, *then* $H(M;p_t^s)=0$ *for all* P_t^s *such that* $s<t$, $s+t\leqslant i+j+1$, $t\geqslant i+1$. *Then*

(i) *If* M *is free over* $B_{i,j,n}$, *then* M *is free over* $B_{i,j+1,n}$.

(ii) *If* $\mathrm{Ext}^{st}_{B_{i,j,n}}(M,\mathbb{Z}_2)=0$ *for* $ds>t+c$ *with* $d\geqslant(2^{i+1}-1)2^j$, *then* $\mathrm{Ext}^{st}_{B_{i,j+1,n}}\times$ $\times(M,\mathbb{Z}_2)=0$ *for* $ds>t+c+(2^{i+1}-1)2^j$.

Proof. If $j\geqslant i+1$, then Theorem 2.7 shows $[\xi_{i+1}^{2j}]$ is nilpotent in $H^*(B_{i,j+1,n})$ and the lemma follows from Theorem 3.2.

If $j<i+1$, then (since $[P_{t_1}^{s_1},P_{t_2}^{s_2}{}_0]=0$ whenever $s_1,s_2<t_1,t_2$) $B_{i,j+1,i+j}$ is an exterior algebra on generators of distinct degree with respect to which M is acyclic. By [4; Theorem 2.1] M is free over $B_{i,j+1,i+j}$. Hence by Proposition 3.4 $\mathrm{Ext}^{st}_{B_{i,j+1,n}}$ $(M,\mathbb{Z}_2)=0$ for $t<2^j(2^{i+2}-1)s$. Thus by Proposition 3.3 $[\xi_{i+1}^{2j}]$ acts nilpotently on $\mathrm{Ext}_{B_{i,j+1,n}}(M,\mathbb{Z}_2)$. The lemma now follows from Theorem 3.2.

Proof of Theorem 1.2. We shall show that if $A_n=A(n+1,n,\ldots,1)$, then M is a free A_n-module. Hence by Proposition 3.4, $\mathrm{Ext}^{s,t}_{\mathscr{A}}(M,\mathbb{Z}_2)=0$ for $2^{n+1}s>t$. Since this is true for all n, $\mathrm{Ext}^{s,t}_{\mathscr{A}}(M,\mathbb{Z}_2)=0$ for $s>0$. Thus M is $\mathscr{A}$-free.

If $n=2m$, then [4; Theorem 2.1] implies M free over the exterior algebra $B_{m,m+1,2m}$. Repeated application of Lemma 4.1 shows "M free over $B_{i,2m+1-i,2m}=B_{i-1,0,2m}$ implies M free over $B_{i-1,2m+1-(i-1),2m}$." Thus by induction M is free over $B_{0,2m+1,2m}=A_{2m}$.

If $n=2m-1$, M is free over the exterior algebra $B_{m-1,m,2m-1}$ and by Lemma 4.1 it is free over $B_{m-1,m+1,2m-1}$. By the same induction as above it is free over $B_{0,2m,2m-1}=A_{2m-1}$. ∎

We can now use exactly the same method to prove a vanishing theorem similar to Theorem 1.1 but with slightly larger value for the y-intercept $c/(d-1)$. In the next section we will consider several methods of decreasing the y-intercept. The result proved here will be necessary to prove the stronger result.

THEOREM 4.2. *If* M *is an* $\mathscr{A}$*-module and* $P_{t_0}^{s_0}$ *is the lowest degree* P_t^s *with* $s<t$ *such that* $H(M;P_t^s)\neq0$, *then* $\mathrm{Ext}^{st}_{\mathscr{A}}(M,\mathbb{Z}_2)=0$ *for* $ds>t+c$, *where* $d=\deg P_{t_0}^{s_0}=2^{s_0}(2^{t_0}-1)$ *and* c *is defined as follows. If* $s_0+t_0=2m+1$, *then* $c=(m-1)2^{2m+2}+2^{m+1}+m+2$. *If* $s_0+t_0=2m$, *then* $c=(m-3/2)2^{2m+1}+2^{m+1}+m+1$. *Thus* $c/(d-1)$ *is approximately* s_0+t_0.

Proof. Suppose $s_0+t_0=2m+1$. By Corollary 3.5, $\mathrm{Ext}^{st}_{B_{m,m+1,2m}}(M,\mathbb{Z}_2)=0$ for $ds>t$. Repeated application of Lemma 4.1 shows "If $\mathrm{Ext}^{st}_{B_{i,2m+1-i,2m}}(M,\mathbb{Z}_2)=0$ for $ds>t+c$, then $\mathrm{Ext}^{st}_{B_{i-1,2m+1-(i-1),2m}}(M,\mathbb{Z}_2)=0$ for $ds>t+c+(2^i-1)$ $(2^{2m+2-i}-1)$." Thus by induction $\mathrm{Ext}^{st}_{B_{0,2m+1,2m}}(M,\mathbb{Z}_2)=0$ for $ds>t+\sum_{i=1}^m(2^i-1)$ $(2^{2m+2-i}-1)$. The theorem now follows from Proposition 3.6.

If $s_0+t_0=2m$, then Corollary 3.5 shows $\mathrm{Ext}^{st}_{B_{m-1,m,2m-1}}(M,\mathbb{Z}_2)=0$ for $ds>$t. Lemma 4.1 shows $\mathrm{Ext}^{st}_{B_{m-1,m+1,2m-1}}(M,\mathbb{Z}_2)=0$ for $ds>t+(2^m-1)2^m$. Proceeding as above, we find $\mathrm{Ext}^{st}_{B_{0,2m,2m-1}}(M, \mathbb{Z}_2)=0$ for $ds>t+(2^m-1)2^m+\sum_{i=1}^{m-1}(2^i-1)(2^{m2+1-i}-1)$. The theorem follows from Proposition 3.6. ∎

Although we make no claims about our y-intercept being minimal, the slope is certainly minimal. Indeed,

THEOREM 4.3. *If A is an algebra over $\mathbb{Z}_2$, and $x\in A_d$ satisfies $x^2=0$ and $H(M;x)\neq 0$, and if $D>d$ and C is arbitrary, then there exists s, t such that $Ds>t+C$ and* $\mathrm{Ext}^{st}_A(M, \mathbb{Z}_2)\neq 0$.

Proof. If

$$\begin{array}{ccccccccc}
0 \leftarrow M \leftarrow & C_0 & \leftarrow & C_1 & \leftarrow & C_2 & \leftarrow\cdots \\
& \nwarrow & \swarrow\nwarrow & & \swarrow & & \\
& & K_0 & & K_1 & & \\
& \swarrow & \searrow\swarrow & & \nwarrow & & \\
& 0 & & 0 & & 0 &
\end{array}$$

is a minimal A-resolution, then the x-homology groups satisfy

$$0\neq H_j(M)\approx H_{j+d}(K_0)\approx\cdots\approx H_{j+sd}(K_{s-1}).$$

Thus C_s has an element in degree $\leqslant j+sd$. Thus for all s there is a $t\leqslant j+sd$ such that $\mathrm{Ext}^{st}_A(M, \mathbb{Z}_2)\neq$;. For large s $j+sd\leqslant -c+sD$. ∎

5. Lowering the Vanishing Line

In order to see the problems involved in lowering the vanishing line, let us consider an example. Suppose the first P^s_t with respect to which M has homology is P^1_3 of degree 14 (so M is a free $\mathscr{A}_2$-module). Theorem 4.2 showed that $\mathrm{Ext}^{st}_{\mathscr{A}}(M, \mathbb{Z}_2)=0$ for $14s>t+12+1+2+4+8$ by building up the algebras in the order $A(0,2,2,1)$, $A(0,3,2,1)$, $A(1,3,2,1)$, $A(2,3,2,1)$, $A(3,3,2,1)$, $A(4,3,2,1)$, $\mathscr{A}$. Since M is free over $A(3,2,1)$, the increments of c when adding the generators P^0_1, P^1_1, and P^2_1 may have seemed somewhat unnecessary. Indeed Proposition 3.4 shows that $\mathrm{Ext}^{st}_{A(3,2,2,1)}(M, \mathbb{Z}_2)=0$ for $14s>t$ so that if we can build the algebras in the order $A(3,2,2,1)$, $A(3,3,2,1)$, $A(4,3,2,1)$, $\mathscr{A}$, we would show $\mathrm{Ext}^{st}_{\mathscr{A}}(M, \mathbb{Z}_2)=0$ for $14s>t+20$. The only problem is the addition of the generator P^2_2, for $[\zeta^4_2]$ may not be nilpotent in $H^*(A(3,3,2,1))$ despite the fact that it was nilpotent in $H^*(A(0,3,2,1))$. Nor does the argument used before to show that elements act nilpotently apply, for P^2_2 need not lie in a subalgebra of $A(3,3,2,1)$ over which M is free and which contains all of $A(3,3,2,1)$ through $\deg(P^2_2)$. However, we are saved by Theorem 4.2, for it gives us a vanishing line for $\mathrm{Ext}_{A(3,3,2,1)}(M, \mathbb{Z}_2)$ which with Proposition 3.3 enables us to conclude that $[\zeta^4_2]$ acts nilpotently. Thus we can prove Theorem 1.1, where the precise value of c is $(t_0-2)2^{t_0+s_0}+2^{s_0+1}$.

Proof. Let $n=t_0+s_0$. Let $B_i=A(n-1, n-2,\ldots, i+2, i+1, i+1, i,\ldots, 1)$. By Theorem 1.2 M is free over $A(n-1, n-2,\ldots, 1)$, and by Proposition 3.4 $\mathrm{Ext}^{st}_{B_{s_0}}(M, \mathbb{Z}_2)=0$ for $ds>t$. Suppose $\mathrm{Ext}^{st}_{B_i}(M, \mathbb{Z}_2)=0$ for

$$ds > t + c_i (i \geqslant s_0) \quad \text{with} \quad c_i = \sum_{j=s_0+1}^{i} 2^j (2^{n-j} - 1).$$

Since $2^{i+1}(2^{n-i-1}-1)\, 2^{s_0}(2^{t_0}-1)$, Theorem 4.2 and Proposition 3.3 apply to show $[\zeta^{2^{i+1}}_{n-i-1}]$ acts nilpotently in $\mathrm{Ext}_{B_{i+1}}(M, \mathbb{Z}_2)$ and Theorem 3.2 then shows $\mathrm{Ext}^{st}_{B_{i+1}}(M, Z_2)=0$ for $ds>t+c_i+2^{i+1}(2^{n-i-1}-1)$, extending the induction hypothesis. Hence $\mathrm{Ext}^{st}_{A(n, n-1,\ldots, 1)}(M, Z_2)=\mathrm{Ext}^{st}_{B_{n-1}}(M, Z_2)=0$ for $ds>t+c_{n-1}$ where

$$\begin{aligned} c_{n-1} &= \sum_{j=s_0+1}^{n-1} (2^n - 2^j) = (n - s_0 - 1)\, 2^n - 2^{s_0+1} (1 + \cdots + 2^{n-s_0-2}) \\ &= (t_0 - 2)\, 2^n + 2^{s_0+1}. \end{aligned}$$

Since $2^n>2^{s_0}(2^{t_0}-1)$, Proposition 3.6 applies to give the desired result for $\mathrm{Ext}(M, Z_2)$. ∎

For small values of t we can get better results by sacrificing slope for intercept. For example, suppose the first nonacyclic P^s_t is P^0_5 of degree 31. M is free over $A(4, 3, 2, 1)$. Hence by Proposition 3.4 $\mathrm{Ext}^{st}_{A(5,4,3,2,1)}(M, \mathbb{Z}_2)=0$ for $16s>t$ and $\mathrm{Ext}^{st}_{A(4,4,3,2,1)}(M, \mathbb{Z}_2)=0$ for $24s>t$. Thus by Theorem 2.7 and Theorem 3.2 applied to $[\xi_1^{16}]$ and Proposition 3.6, $\mathrm{Ext}^{st}_{\mathscr{A}}(M, \mathbb{Z}_2)=0$ if $16s>t$ or if $24s>t+16$. Similarly Proposition 3.4 shows $\mathrm{Ext}^{st}_{A(4,3,3,2,1)}(M, \mathbb{Z}_2)=0$ for $28s>t$ and hence two applications of Theorem 3.2 to build it up to $A(5, 4, 3, 2, 1)$ followed by 3.6 shows $\mathrm{Ext}^{st}_{\mathscr{A}}(M, \mathbb{Z}_2)=0$ for $28s>t+40$. Thus we see that for such modules M $\mathrm{Ext}^{st}_{\mathscr{A}}(M, \mathbb{Z}_2)=0$ for $t>\max(16s, 24s-16, 28s-40, 30s-68)$.

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