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Integral Domains with Noetherian Subrings

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Let D be an integral domain with identity having quotient field K and prime subring Π. The purpose of this paper is to determine necessary and sufficient conditions in order that each subring of D with identity be Noetherian. Such conditions are given in the following theorem.

THEOREM. *If $\Pi = Z$, the ring of integers, then each subring of D with identity is Noetherian if and only if $[K:Q] < \infty$, where Q denotes the field of rational numbers. If $\Pi = GF(p)$, then each subring of D with identity is Noetherian if and only if either* (1) *K/Π is algebraic, or* (2) *K is a finite algebraic extension of a purely transcendental extension of Π of transcendence degree* 1.

In order to avoid unnecessary duplication in the cases when $\Pi = Z$ and when $\Pi = GF(p)$, we proceed through a series of general results which will prove to be pertinent to both of the above cases. As a case in point, both Z and $GF(p)[X]$ are Dedekind domains in which each nonzero ideal has finite index, so that a theorem proved about such Dedekind domains will apply to both Z and $GF(p)[X]$. Each ring considered in this paper is assumed to be commutative.

THEOREM 1. *Suppose that R is a commutative ring with identity which is not its own total quotient ring, and that X is an indeterminate over R. There is a non-Noetherian subring of $R[X]$ with identity.*

Proof. Let e be the identity element of R. There is a regular element t of R which is not a unit of R. Thus if $A = (t)$, then $A \subset R$ and the only element x of R such that $tx = t$ is $x = e$, so that A is a ring without identity. By [5], $A[X]$ is not Noetherian, and by [6, p. 184], the subring of $R[X]$ generated by $A[X]$ and e is also non-Noetherian.

COROLLARY 1. *If D is an integral domain with identity and if X and Y are indeterminates over D, then $D[X, Y]$ contains a non-Noetherian subring with identity; if D is not a field, $D[X]$ contains a non-Noetherian subring with identity.*

COROLLARY 2. *Suppose that D is an integral domain with identity having quotient field K and prime subring Π, and suppose that each subring of D with identity is Noetherian. If $\Pi = Z$, then K is algebraic over Q. If $\Pi = GF(p)$, then* tr.d. $K/\Pi \leqslant 1$.

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Proof. This is immediate from Corollary 1 and from the fact that D contains a transcendence basis for K over the quotient field of Π [16, p. 99].

LEMMA 1. *Suppose that J is an integral domain with identity which is algebraic over J_0, a subring of J. If J' is the integral closure of J_0 in J, then J and J' have the same quotient field.*

Proof. It suffices to observe that J is contained in the quotient field of J'. Thus is $x \in J$, then x is algebraic over J_0 so that dx is integral over J_0 for some nonzero element d of $J_0 \subseteq J'$ [8, p. 78]. Therefore, $x = dx/d$ belongs to the quotient field of J'.

Our next result will be obtained as a corollary to a theorem due to Arnold and Gilmer [1, p. 142], and will use the following notation: Let D_0 be a Dedekind domain with quotient field K_0, let L be an infinite-dimensional separable algebraic extension field of K_0 which is expressed as the union of a net $\{K_\alpha\}_{\alpha \in A}$ of finite algebraic extension fields of K_0 (here *net* means that for any α, $\beta \in A$, there exists $\delta \in A$ such that K_α and K_β are contained in K_δ), let D be the integral closure of D_0 in L, and for each α in A, let D_α be the integral closure of D_0 in K_α. The theorem of Arnold and Gilmer referred to states that D is a Dedekind domain if and only if for each prime ideal P_0 of D_0, there is an element $\alpha(P_0)$ in A such that each prime ideal of $D_{\alpha(P_0)}$ lying over P_0 in D_0 is inertial with respect to D. The corollary to this theorem which we need is

COROLLARY 3. *If D is a Dedekind domain, then D is the only ideal of D having finite index in D.*

Proof. It suffices to show that D/M is infinite for each maximal ideal M of D. Thus, let $P_0 = M \cap D_0$ and choose α in A such that each prime ideal of D_α lying over P_0 in D_0 is inertial with respect to D. Set $P = M \cap D_\alpha$. For any β in A such that $D_\alpha \subseteq D_\beta$, PD_β is the unique maximal ideal of D_β lying over P in D_α. Since $PD_\beta \subseteq M \cap D_\beta \subset D_\beta$ for each such β, it then follows that $M \cap D_\beta = PD_\beta$ for each such β. Now fix a positive integer N. We show that $|D/M| > N$ by showing that $[D/M : D_\alpha/P]$, the dimension of D/M as a vector space over D_α/P, is greater than N. Since $[L:K_0] = \infty$ and since $[K_\alpha : K] < \infty$, there exists $\beta \in A$ such that $K_\alpha \subseteq K_\beta$ and such that $[K_\beta : K_\alpha] > N$. Since K_β/K_α is separable, a well-known theorem due to Roquette [15] and to Cohen and Zariski [4] then implies that $[D_\beta/PD_\beta : D_\alpha/P] = [K_\beta : K_\alpha] > N$. However, $D_\alpha/P \subseteq D_\beta/PD_\beta \subseteq D/M$, so that $[D/M : D_\alpha/P] > N$ also.

THEOREM 2. *Suppose that D is an integral domain with identity such that D/A has nonzero characteristic for each nonzero ideal A of D. If each subring of D with identity is Noetherian, then D/A is finite for each nonzero ideal A of D.*

Proof. If A is a nonzero ideal of D and if e is the identity element of D, then by hypothesis, the subring A^* of D generated by A and e is Noetherian. Hence A, considered as a ring, is Noetherian. A theorem due to Butts and Gilbert [2, Theorem 6] then implies that D is a finite A^*-module, so that D/A is a finite A^*/A-module. Now $A^*/A \simeq \Pi/(\Pi \cap A)$, where Π is the prime subring of D, and the hypothesis on D implies that $\Pi/(\Pi \cap A)$ is a finite ring. Hence A^*/A is finite and D/A is also finite.

THEOREM 3. *Suppose that D is a domain with identity having quotient field K and prime subring Z; we assume that K/Q is algebraic. If D' is the integral closure of Z in D and if Z' is the integral closure of Z in K, then these conditions are equivalent*:

1) *Each subring of D with identity is Noetherian.*
2) *Each subring of D' with identity is Noetherian.*
3) *Z' is a Dedekind domain, and each nonzero ideal of Z' has finite index in Z'.*
4) $[K:Q]<\infty$.
5) *Each subring of K with identity is Noetherian.*

Proof. 1)→2): clear.

2)→3): Lemma 1 shows that D' has quotient field K, and since D' is integral over Z, D' is one-dimensional. By what Nagata calls the Krull-Akizuki Theorem in [14, p. 115] (see also [3, p. 29]), it then follows that Z' is one-dimensional Noetherian and that Z'/A' is a finite $D'/(A' \cap D')$-module for each nonzero ideal A' of Z'. Since Z' is integrally closed, Z' is therefore a Dedekind domain. We observe that since D' is integral over Z, each nonzero ideal of D' meets Z in a nonzero ideal of Z so that D'/B has finite characteristic for each nonzero ideal B of D'. By Theorem 2, each such D'/B is finite; in particular, $D'/(A' \cap D')$ is finite for each nonzero ideal A' of Z' so that Z'/A' is also finite.

3)→4): This is immediate from Corollary 3 once we observe that the family $\{K_\alpha\}_{\alpha \in A}$ of all subfields of K which are finite-dimensional over Q form a net with union K.

4)→5): Apply the Krull-Akizuki Theorem.

5)→1): Clear.

If K is an algebraic extension field of $GF(p)$, then each subring of K with identity is a field [17, p. 12], and hence is Noetherian. Therefore in characterizing domains of nonzero characteristic for which every subring with identity is Noetherian, we can, by Corollary 2, reduce to the case where the transcendence degree of the quotient field of the domain over its prime subfield is 1.

THEOREM 4. *Suppose that D is a domain with identity having quotient field K and prime subring $\Pi = GF(p)$; we assume that* tr.d. $K/\Pi = 1$ *and that X is an element of D transcendental over Π. Let $J = \Pi[X]$, let D' be the integral closure of J in D, and let J' be the integral closure of J in K. Then these conditions are equivalent;*

1) *Each subring of D with identity is Noetherian.*
2) *Each subring of D' with identity is Noetherian.*
3) *J' is a Dedekind domain, and each nonzero ideal of J' has finite index in J'.*

4) $[K:\Pi(X)]<\infty$.

5) *Each subring of K with identity is Noetherian.*

Proof. As in the proof of Theorem 3, the implications 1)→2) and 5)→1) are clear. The proof that 2) implies 3) is the same as that given in Theorem 3 once we observe that $J=\Pi[X]$ is one-dimensional and that D'/B has characteristic p for each genuine ideal B of D'.

3)→4): For this part of Theorem 4 we shall need the following result due to Gilmer, Heinzer, and Kreimer [9]:

Suppose that F is a field of characteristic $p>0$ *and that F is not separably algebraically closed. These conditions are equivalent.*

a) $F^{1/p}$ *is a simple extension of F.*

b) *There are no exceptional extensions of F. That is, if K is an inseparable extension of F, then the purely inseparable part of K over F properly contains F.*

c) *Each algebraic extension of F splits over F. That is, if K is an algebraic extension field of F and if* K_i *is the purely inseparable part of K over F, then* K/K_i *is separable.*

d) *Each finite-dimensional extension of F is simple over F.*

To apply this theorem to our problem, we observe that $[\Pi(X)]^{1/p}=\Pi(X^{1/p})=$ $=\Pi(X)(X^{1/p})$ since Π is a perfect field. Hence the result of Gilmer, Heinzer, and Kreimer cited above shows that if K_i is the purely inseparable part of $K/\Pi(X)$, then K is separable over K_i. We let J_i be the integral closure of J in K_i. Since J', the integral closure of J_i in K, is a Dedekind domain in which each nonzero ideal has finite index, then J_i is a Dedekind domain [10, p. 750], and since K/K_i is separable, Corollary 3 implies that $[K:K_i]$ is finite. Finally, since $AJ'\cap J_i=A$ for each ideal A of J_i [7, p. 563], J_i/A is isomorphic to a subring of J'/AJ', and consequently, J_i/A is finite for each nonzero ideal A of J_i. To prove that $[K:\Pi(X)]$ is finite, we now need only to show that $[K_i:\Pi(X)]$ is finite; that is, we can reduce to the case when $K/\Pi(X)$ is purely inseparable.

It is known (see footnote 5 of [13]) that any finite-dimensional purely inseparable extension of $\Pi(X)$ is of the form $\Pi(X^{1/p^e})$ for some positive integer e, so that $\Pi(\{X^{1/p^e}\}_{e=1}^{\infty})$ is the only purely inseparable infinite-dimensional extension field of $\Pi(X)$. However, $T=\Pi[\{X^{1/p^e}\}_{e=1}^{\infty}]$ is the integral closure of $\Pi[X]$ in $\Pi(\{X^{1/p^e}\}_1^{\infty})$, and T is not a Dedekind domain since $\{X^{1/p^e}\}_1^{\infty}T$ is an idempotent maximal ideal of T. Therefore, since J_i is a Dedekind domain, $[K_i:\Pi(X)]$ is finite, and 4) holds.

4)→5): Consider a subring R of K with identity, and let L be the quotient field of R. If L/Π is algebraic, we have already observed that each subring of L is Noetherian. If L/Π is not algebraic, then L has transcendence degree 1 over Π and we can choose Y in R such that Y is transcendental over Π. Since tr.d. $K/\Pi=1$, K is algebraic over $\Pi(Y)$. Therefore, $[\Pi(Y,X):\Pi(Y)]<\infty$, and since $[K:\Pi(X)]<\infty$, $[K:\Pi(Y,X)]<\infty$ also. We conclude that $[K:\Pi(Y)]=[K:\Pi(Y,X)]\,[\Pi(Y,X):\Pi(Y)]$ is finite, and since $\Pi(Y)\subseteq L\subseteq K$, $[L:\Pi(Y)]$ is also finite. Applying the Krull-Akizuki theorem to

$\Pi[Y]$, R, $\Pi(Y)$, and L, we then conclude that R is Noetherian, and our proof that 4) implies 5) is complete.

Remark. If R is a ring with identity e and if each subring of R with identity is Noetherian, then each subring of R is Noetherian, for if S is any subring of R, then $S[e]$ is Noetherian, and as previously mentioned, this implies that S is Noetherian.

We observe that our proofs of Theorems 3 and 4 show that if each subring of a domain D with identity is Noetherian, then each subring of D with identity has dimension $\leqslant 1$. Using the notion of *valuative dimension* introduced by Jaffard in [11] (see also [12, Chap. 4] and [8, § 25]), it is easy to characterize domains D with identity having what we shall designate here as property (d_n), where n is a positive integer.

(d_n): *Each subring of D with identity has dimension* $\leqslant n$.

In fact, we have

THEOREM 5. *Let D be an integral domain with identity having quotient field K and prime subring* Π.

a) *If* $\Pi = Z$, *then D has property* (d_n) *if and only if* tr.d. $(K/Q) \leqslant n-1$.

b) *If* $\Pi = GF(p)$, *then D has property* (d_n) *if and only if* tr.d. $(K/\Pi) \leqslant n$.

Proof. If R is a commutative ring with identity and if $X_1, \ldots, X_k$ are indeterminates over R, then $\dim R[X_1, \ldots, X_k] \geqslant \dim R + k$, so that if D has property (d_n), then tr.d. $(K/Q) \leqslant n-1$ in case a) and tr.d. $(K/Q) \leqslant n$ in case b).

To prove the converses, let J be any subring of K and let L be the quotient field of J. To show that $\dim J \leqslant n$, it suffices to show that the valuative dimension of J is at most n [8, p. 211]. Thus if V is a valuation ring on L containing J, then in case a), $\dim V \leqslant \dim(V \cap Q) + \text{tr.d.}(L/Q) \leqslant \dim(V \cap Q) + n - 1 \leqslant n$ (see [12, p. 10] or [8, p. 239]), the last inequality holding because any valuation ring on Q has rank at most 1. And in case b), $\dim V \leqslant \dim(V \cap \Pi) + \text{tr.d.}(L/Q) = \text{tr.d.}(L/Q) \leqslant n$. Consequently, in case a) or case b), the valuative dimension of J is at most n, and hence the dimension of J is at most n.

Remark. If S is a commutative ring with identity e, if R is a subring of S, and if R^* is the subring of S generated by R and e, then it is known that $\dim R \leqslant \dim R^* \leqslant \dim R + 2$. Hence if each subring of S with identity has dimension $\leqslant n$, then each subring of S has dimension $\leqslant n$.

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