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Collars and Concordances of Topological Manifolds

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Let $F: M \times [0, 1] \to E \times [0, 1]$ be a concordance of a compact topological m-manifold in Euclidean q-space. It will be shown that there is an ambient isotopy H of $E \times [0, 1]$, fixed on $E \times 0$, such that $H_1 F$ is vertical, provided $m \leqslant q-3$. In particular then, concordant embeddings[1]) of a compact topological manifold in Euclidean space are ambient isotopic if the codimension is at least 3. A direct application sharpens results of Lees [4], [5] and provides an unknotting theorem for topological manifolds in Euclidean space.

The proof is a simple inductive application of Hudson's corresponding piecewise linear result [2] and Miller's approximation theorem [6] over coordinate neighbourhoods. In approximating topological embeddings of bounded piecewise linear manifolds by piecewise linear embeddings, extension and uniqueness theorems for collars of topological manifolds will be used. These results on collars are isolated in the first section.

Collars of Manifolds

Let Q be a topological manifold. A *collar* of Q will be a (locally flat) embedding

$$c: \partial Q \times [0, 1] \to Q$$

such that $c(x, 0) = x$ for all $x \in \partial Q$. Every topological manifold has a collar by Brown [1]. (Note that Brown deals with open collars, i.e. embeddings $c: \partial Q \times [0, 1) \to Q$ such that $(x, 0) \mapsto x$ for all $x \in \partial Q$. Clearly the restriction of such an open collar to $\partial Q \times [0, \frac{1}{2}]$ gives a "collar" when suitably reparametrized.)

THEOREM 1. *Let Q be a topological manifold, M a compact submanifold of ∂Q and $\bar{e}: M \times [0, 2] \to Q$ an embedding such that $\bar{e}(x, 0) = x$ for all $x \in M$ and $\bar{e}^{-1}\, \partial Q = M \times 0$. Let $e = \bar{e} \mid M \times [0, 1]$. Then there is a collar c of Q satisfying $c \mid M \times [0, 1] = e$.*

COROLLARY. *Let $N \subset Q$ be topological manifolds with ∂N compact and $N \cap \partial Q = \partial N$. Then any collar of N can be extended to a collar of Q. (i.e. given a collar d of N there is a collar c of Q such that $c \mid \partial N \times [0, 1] = d$.)*

Recall that an *ambient isotopy* of Q is a level preserving homeomorphism $H: Q \times [0, 1] \to Q \times [0, 1]$ for which H_0 is the identity. (Level preserving means that H

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1) Throughout this note all embeddings are assumed locally flat.

maps $Q \times t$ to $Q \times t$ for each $t \in [0, 1]$; the homeomorphism $H_t: Q \to Q$ is defined by $H \mid Q \times t = H_t \times 1$.) The ambient isotopy H is *fixed* on the subset X of Q if $H_t(x) = x$ for all $x \in X$ and all $t \in [0, 1]$.

Define a shortened embedding

$$e_\varepsilon: M \times [0, 1] \to Q, \quad 0 < \varepsilon < 1, \quad \text{by} \quad e_\varepsilon(x, t) = e(x, \varepsilon t)$$

and observe:

LEMMA 1. *There is an ambient isotopy H of Q, fixed on ∂Q, such that $H_1 e = e_\varepsilon$.*

Proof. For $0 \leqslant u \leqslant 1$ let

$$\lambda_u: [0, 2] \to [0, 2]$$

be the map specified by

$$0 \to 0, \quad 1 \to 1 - (1 - \varepsilon)\, u, \quad 2 \to 2$$

and extending linearly. Define an ambient isotopy K of $M \times [0, 2]$ by

$$K_u(x, t) = (x, \lambda_u(t)), \quad 0 \leqslant u \leqslant 1,$$

for all $x \in M$, $t \in [0, 2]$. Note that it keeps $(M \times 0) \cup (M \times 2)$ fixed and shrinks $M \times [0, 1]$ to $M \times [0, \varepsilon]$. It is not difficult to show that K_1 can be written as the composition $k^r \cdots k^2 k^1$ of a finite number of homeomorphisms each of which keeps $(M \times 0) \cup \cup (M \times 2)$ fixed and has a ball for support. Moreover, these supporting balls can be taken to have arbitrarily small diameter. Consequently, by the local flatness of $\bar{e}$, the homeomorphism g^i of $\bar{e}(M \times [0, 2])$ defined by

$$z \mapsto \bar{e} k^i \bar{e}^{-1}(z)$$

can be extended to a homeomorphism h^i of Q that keeps ∂Q fixed and has a ball for support; in particular h^i is isotopic to the identity keeping ∂Q fixed. Then the composite isotopy H of $h^r \ldots h^2 h^1$ to the identity will satisfy $H_1 e = e_\varepsilon$ as required.

LEMMA 2. *Given $x \in M$ there is a neighbourhood U of x in ∂Q, a real number $\varepsilon > 0$ and an embedding*

$$d: U \times [0, 1] \to Q$$

such that

$$d^{-1}\, \partial Q = U \times 0,$$

and

$$d \,|(U \cap M) \times [0, 1] = e_\varepsilon | .$$

Proof. Suppose x lies in the interior of M. Now M is locally flat in ∂Q, so a neighbourhood pair of x in $M \subset \partial Q$ can be identified with a standard Euclidean pair $E^m \subset E^{q-1}$, x being identified with the origin. Since e is a locally flat embedding there is a chart

$$h: E^{q-1} \times [0, 1] \to Q$$

such that

$$h(x, 0) = x \quad \text{for all} \quad x \in E^{q-1}$$

$$h^{-1}\, \partial Q = E^{q-1} \times 0,$$

and

$$h^{-1} e(M \times [0, 1]) = E^m \times [0, 1].$$

Let B^m denote the unit ball in E^m and choose $\varepsilon > 0$ so small that $h^{-1} e_\varepsilon$ embeds $B^m \times [0, 1]$ in $E^m \times [0, 1]$. Regard E^{q-1} as the product $E^m \times E^{q-m-1}$ and extend $h^{-1} e_\varepsilon \mid B^m \times [0, 1]$ productwise to an embedding f of $B^m \times E^{q-m-1} \times [0, 1]$ in $E^{q-1} \times [0, 1]$. Then $B^m \times E^{q-m-1}$ will do for the neighbourhood U, and hf for the embedding d. The case $x \in \partial M$ is dealt with in a similar manner.

Proof of Theorem 1. The method of proof is due to Brown [1] and involves knitting together the local extensions of Lemma 2; minor modifications are necessary in order to deal with closed collars.

First produce a neighbourhood N of M in ∂Q, a real number $\delta > 0$ and an embedding $h: N \times [0, 1] \to Q$ such that $h^{-1}\, \partial Q = N \times 0$ and $h \mid M \times [0, 1] = e_\delta$. This is done as follows. For each $x \in M$ choose a neighbourhood U of x in ∂Q of the type provided by Lemma 2. Let V be a closed neighbourhood of x that lies in the interior of U. Since M is compact, a finite number $U_1 \ldots U_r$ of such neighbourhoods can be selected so that $M \subset V_1 \cup \cdots \cup V_r$. Proceed by induction, producing at stage k a neighbourhood N_k of $A_k = V_1 \cup \cdots \cup V_k$ in ∂Q, a real number $\delta_k > 0$ and an embedding $h_k: N_k \times [0, 1] \to Q$ such that $h_k^{-1}\, \partial Q = N_k \times 0$ and

$$h_k | (N_k \cap M) \times [0, 1] = e_{\delta_k} | .$$

When $k = r$, set $N = N_r$, $\delta = \delta_r$ and $h = h_r$.‘

The induction begins with $k = 1$ by Lemma 2. Consider the inductive step k to $k+1$. By hypothesis N_k, δ_k, h_k exist with the above properties. In addition, by the choice of the U's, there exists a real number ε, $0 < \varepsilon \leqslant \delta_k$, and an embedding $d: U_{k+1} \times [0, 1] \to Q$ such that

(i) $d^{-1}\, \partial Q = U_{k+1} \times 0$, and

(ii) d cut down to $(U_{k+1} \cap M) \times [0, 1]$ agrees with e_ε.

Note that $g_k = (h_k)_{\varepsilon/\delta_k}$ agrees with $e_\varepsilon on (N_k \cap M) \times [0, 1]$. Using Brown's Lemmas 2 and 3, d can be replaced by a new embedding d' which satisfies (i) and (ii) above and in addition agrees with g_k on some neighbourhood W_0 of $A_k \cap V_{k+1}$ in $(N_k \cap U_{k+1}) \times \times [0, 1]$. The argument will now be completed following Brown Lemma 4. Choose disjoint open subsets O_1, O_2 of Q such that

$$A_k - V_{k+1} \subset O_1 \subset g_k(N_k \times [0, 1]),$$

$$V_{k+1} - A_k \subset O_2 \subset d'(U_{k+1} \times [0, 1]).$$

Next choose disjoint open subsets W_1, W_2 of $(N_k \cup U_{k+1}) \times [0, 1]$ such that

$$A_k - V_{k+1} \subset W_1 \subset g_k^{-1} O_1,$$

$$V_{k+1} - A_k \subset W_2 \subset (d')^{-1} O_2.$$

Let $W = W_0 \cup W_1 \cup W_2$: then W is a neighbourhood of A_{k+1} in $(N_k \cup U_{k+1}) \times [0, 1]$. Define an embedding f of W in Q by

$$f = \begin{cases} g_k = d' & \text{on} \quad W_0, \\ g_k & \text{on} \quad W_1, \\ d' & \text{on} \quad W_2. \end{cases}$$

Let N_{k+1} be a closed neighbourhood of A_{k+1} in ∂Q that lies in the interior of W, and choose $\delta_{k+1} > 0$ so small that

$$N_{k+1} \times [0, 2\delta_{k+1}] \subset W.$$

Let $s: N_{k+1} \times [0, 1] \to N_{k+1} \times [0, 2\delta_{k+1}]$ be a fibre preserving shrinking homeomorphism which is the identity on $N_{k+1} \times [0, \delta_{k+1}]$. Finally, define h_{k+1} to be the composition fs. This completes the inductive step.

Now extend $h: N \times [0, 1] \to Q$ over the remainder of ∂Q as follows. Let N_* be a closed neighbourhood of M in ∂Q which lies in the interior of N. Choose an arbitrary collar of $Q - h(N_* \times [0, 1])$ and use Brown's procedure again to knit this collar to h, so producing a collar say c_* of Q. The details will not be repeated; let it suffice to say that the necessary modifications occur away from $e(M \times [0, 1])$ with the exception of the final shrinking process. Therefore $c_* \mid M \times [0, 1] = e_\varepsilon$ for some ε satisfying $0 < \varepsilon \leqslant \delta$.

Finally, by Lemma 1, there is an ambient isotopy H of Q, which is the identity on ∂Q, such that $H_1 e = e_\varepsilon$. Then the collar $c = H_1^{-1} c_*$ of Q satisfies $c \mid M \times [0, 1] = e$. This completes the proof.

Proof of the Corollary. Since d is a locally flat embedding, $N - d(\partial N \times [0, 1))$ is a manifold. A collar of this manifold will extend d to $\bar{d}: \partial N \times [0, 2] \to N \subset Q$. Now apply the theorem with $M = \partial N$ and $e = d$.

THEOREM 2. *Any two collars of a topological manifold are ambient isotopic keeping the boundary fixed.*

Remarks. 1. This result is not new. The argument given here (slide down one collar, then slide back up the other) was pointed out to me by *Professor R. K. Lashof.*

2. The theorem is not true if the hypothesis of local flatness is omitted from the definition of a collar. For example, let B_a be the ball $\{x \in E^3 \mid \|x\| \leqslant a\}$, and let $f: B_3 \to E^3$ be an embedding that sends ∂B_3 onto a sphere that is not bicollared. Then $f \mid B_2 - \operatorname{int} B_1$ and $f \mid B_3 - \operatorname{int} B_1$ are both collars of $E^3 - f(\operatorname{int} B_1)$, but they are certainly not ambient isotopic.

Proof of Theorem 2. Let $c, d: \partial Q \times [0, 1] \to Q$ be collars of the topological manifold Q. Choose a continuous function $\varepsilon(x): \partial Q \to (0, 1]$ so that, if $d_{\varepsilon(x)}$ is the collar of Q defined by

$$d_{\varepsilon(x)}(x, t) = d(x, \varepsilon(x)\, t)$$

then

$$d_{\varepsilon(x)}(\partial Q \times [0, 1]) \subset c(\partial Q \times [0, 1]).$$

Since d is locally flat it extends to an embedding of $\partial Q \times [0, 2]$ in Q. In the usual way d can be ambient isotoped to $d_{\varepsilon(x)}$ by an ambient isotopy of Q that fixes ∂Q and $Q - d(\partial Q \times [0, 2))$.

Now consider

$$k = c^{-1} d_{\varepsilon(x)}: \partial Q \times [0, 1] \to \partial Q \times [0, 1].$$

Suppose there is an ambient isotopy K of $\partial Q \times [0, 1]$, fixed on $(\partial Q \times 0) \cup (\partial Q \times 1)$ such that

$$K_1(x, t) = k(x, t) \quad \text{for all} \quad x \in \partial Q, \quad 0 \leqslant t \leqslant \tfrac{1}{2}.$$

Then defining $H_s: Q \to Q$, $0 \leqslant s \leqslant 1$, by

$$H_s(z) = \begin{cases} c K_s c^{-1}(z) & \text{for} \quad z \in c(\partial Q \times [0, 1]) \\ z & \text{otherwise} \end{cases}$$

gives an ambient isotopy of Q keeping ∂Q fixed such that

$$H_1 c(x, t) = d_{\varepsilon(x)}(x, t)$$

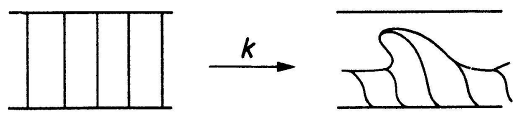
for all $x \in \partial Q$, $0 \leqslant t \leqslant \frac{1}{2}$.

Thus there is a sequence of ambient isotopic collars

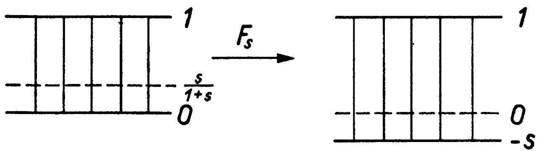
$$c, c_{1/2}, d_{\varepsilon(x)/2}, d.$$

To define K, construct K_s, $0 \leqslant s \leqslant 1$, as a composition

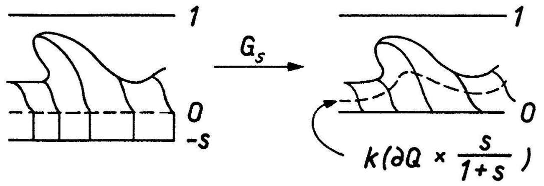
$$\partial Q \times [0, 1] \xrightarrow{F_s} \partial Q \times [-s, 1] \xrightarrow{G_s} \partial Q \times [0, 1]$$



(a) $k:\partial Q \times [0, 1] \to \partial Q \times [0, 1]$



(b) $F_s:\partial Q \times [0, 1] \to \partial Q \times [-s, 1]$: "stretch along the vertical fibres keeping $\partial Q \times 1$ fixed".



(c) $G_s:\partial Q \times [-s, 1] \to \partial Q \times [0, 1]$: "Shrink back along the wiggly fibres keeping the complement of $k(\partial Q \times [0, 1])$ fixed".

where

$$F_s(x, t) = (x, (1+s)\,t - s) \quad \text{for all} \quad x \in \partial Q, \quad t \in [0, 1]$$

and

$$G_s(x, u) = \begin{cases} k\left(x, \dfrac{u+s}{1+s}\right) & \text{for} \quad x \in \partial Q, u \in [-s, 0] \\ k\left(\pi_1 k^{-1}(x, u), \dfrac{\pi_2 k^{-1}(x, u) + s}{1+s}\right) & \text{for} \quad (x, u) \in k(\partial Q \times [0, 1]) \\ (x, u) \quad \text{otherwise}. \end{cases}$$

THEOREM 3. *Let W be a piecewise linear m-manifold, P a compact piecewise linear $(m-1)$-submanifold of ∂W and Q a piecewise linear q-manifold. Let $f: W \to Q$ be an embedding such that $f^{-1}\,\partial Q = P$ and $f \mid P$ is piecewise linear. Then there exists an ambient isotopy H of Q, fixed on ∂Q, such that $H_1 f$ is piecewise linear on a neighbourhood of P in W.*

Proof. Choose an embedding $a: P \times [0, 2] \to W$ onto a neighbourhood of P in W such that

$$a(x, 0) = x \quad \text{for all} \quad x \in P,$$

and

$$a^{-1}\,\partial W = (P \times 0) \cup (\partial P \times [0, 2]).$$

Define an embedding

$$\bar{e}: fP \times [0, 2] \to Q$$

by

$$\bar{e}(x, t) = fa(f^{-1}x, t)$$

for all $x \in fP$, $t \in [0, 2]$. Note that $\bar{e}(x, 0)=x$ for all $x \in fP$ and $\bar{e}^{-1}\,\partial Q = fP \times 0$. Use Theorem 1 to extend $\bar{e} \mid fP \times [0, 1]$ to a collar c of Q. Let d be a piecewise linear collar of Q (i.e. d is a collar and a piecewise linear embedding – see [7] Chapter 5). Theorem 2 provides an ambient isotopy H of Q, fixed on ∂Q, such that $H_1 c = d$. Then $H_1 f: W \to Q$ is piecewise linear on $a(P \times [0, 1])$.

Concordances of Manifolds in Euclidean Space

Let M, Q be topological manifolds and I denote the closed interval $[0, 1]$. Two embeddings f and g of M in Q are *concordant* if there is an embedding $F: M \times I \to Q \times I$ which satisfies

$$F^{-1}(Q \times 0) = M \times 0, \qquad F^{-1}(Q \times 1) = M \times 1,$$
$$F \mid M \times 0 = f, \qquad F \mid M \times 1 = g.$$

The concordance is said to be *vertical* over a subset X of M if $F(x, t) = (f(x), t)$ for all $x \in X$, $t \in I$. Vertical over M will be abbreviated to simply "vertical". The embeddings $f, g: M \to Q$ are *ambient isotopic* if there is an ambient isotopy H of Q such that $H_1 f = g$.

THEOREM 4. *Let M be a compact topological m-manifold and E Euclidean q-space. If $f, g: M \to E$ are concordant via $F: M \times I \to E \times I$, and if $m \leqslant q-3$, then there is an ambient isotopy $\mathscr{H}$ of $E \times I$, fixed on $E \times 0$, such that $\mathscr{H}_1 F$ is vertical.*

COROLLARY. *Concordant embeddings of a compact topological m-manifold in Euclidean q-space are ambient isotopic provided $m \leqslant q-3$.*

Remarks.

1. That Theorem 4 implies the Corollary is seen by restricting $\mathscr{H}$ to $E \times 1$.

2. The method of proof of Theorem 4 shows a little more – extra details will be left to the reader:

(a) If F is given vertical over a neighbourhood of a closed subset X of M, then $\mathscr{H}$ can be taken to be fixed on a neighbourhood of $F(X \times I)$.

(b) E may be replaced by any open piecewise linear q-manifold.

3. Let $f: M \to E$ be an embedding of a closed topological m-manifold in Euclidean q-space. One can define a continuous equivariant map from $\tilde{M} = M \times M - \Delta_M$ to the unit sphere S^{q-1} by

$$\tilde{f}(x, y) = \frac{f(x) - f(y)}{\|f(x) - f(y)\|}.$$

Lees has shown ([4] Theorem 2) that if g is a second embedding with $\tilde{g}$ equivariantly homotopic to $\tilde{f}$, and if $2q > 3(m+1)+1$, $m \geqslant 2$, then f and g are concordant. Using the above Corollary we deduce that f and g are ambient isotopic.

The following two results will be needed. Let W be a compact piecewise linear m-manifold, P a compact piecewise linear $(m-1)$-submanifold of ∂W, and Q a piecewise linear q-manifold. Suppose $m \leqslant q-3$.

THEOREM (Hudson [2], [3]). *Let $F: W \times I \to Q \times I$ be a piecewise linear concordance such that $F^{-1}(\partial Q \times I) = P \times I$ and F is vertical over P. Then there is an ambient isotopy $\mathscr{H}$ of $Q \times I$, fixed on $(Q \times O) \cup (\partial Q \times I)$, such that $\mathscr{H}_1 F$ is vertical.*

THEOREM (Miller [6]). *Let $f: W \to Q$ be a locally flat embedding such that $f^{-1}\, \partial Q = P$ and f is piecewise linear on a neighbourhood of P in W. If $q \geqslant 5$ there is an ambient isotopy H of Q, fixed on ∂Q, such that $H_1 f$ is piecewise linear.*

Proof of Theorem 4. First assume $q \geqslant 5$. Cover M by a finite number of balls $B_1, \ldots, B_r$ each of which is locally flat in M. Proceed by induction and show that, if $A_k = B_1 \cup \cdots \cup B_k$, there is a neighbourhood U_k of A_k in M and an ambient isotopy $\mathscr{H}^k$ of $E \times I$, fixed on $E \times O$, such that $\mathscr{H}^k_1 F$ is vertical over U_k. The proof will then be complete when $k = r$.

The induction *begins* with $k = 1$. Let B be a locally flat ball in M which contains B_1 in its interior (say B_1 plus a collar on its boundary), and regard B as a piecewise linear manifold. By Miller's approximation theorem there are ambient isotopies α and β of $E \times O$, $E \times 1$ respectively such that $\alpha_1 f \mid B$ and $\beta_1 g \mid B$ are piecewise linear embeddings. Combine α and β to give an ambient isotopy $\mathscr{A}$ of $E \times I$ (i.e. $\mathscr{A}$ restricts to α on $E \times O$ and β on $E \times 1$) and let G denote the embedding

$$\mathscr{A}_1 F \mid B \times I : B \times I \to E \times I.$$

It is enough to find an ambient isotopy $\mathscr{G}$ of $E \times I$, which keeps $E \times O$ fixed, such that $\mathscr{G}_1 G$ is vertical. For define an ambient isotopy $\alpha \times 1$ of $E \times I$ by

$$(\alpha \times 1)_u (x, t) = (\alpha_u(x), t),$$

$0 \leqslant u \leqslant 1$, $x \in E$, $t \in I$. If $\mathscr{H}^1$ is the composition: $\mathscr{A}$ followed by $\mathscr{G}$ followed by the inverse of $\alpha \times 1$, then $\mathscr{H}^1$ keeps $E \times O$ fixed and $\mathscr{H}^1_1 F$ is vertical over B.

The ambient isotopy $\mathscr{G}$ will itself be defined as a composition. First apply Theorem 3, obtaining an ambient isotopy of $E\times I$ which keeps $(E\times O)\cup(E\times 1)$ fixed and makes G piecewise linear on a neighbourhood of $(B\times O)\cup(B\times 1)$. Then an application of Miller's theorem allows us to ambient isotop G to a piecewise linear embedding without moving $(E\times O)\cup(E\times 1)$. Finally, by Hudson, there is an ambient isotopy of $E\times I$ which keeps $E\times O$ fixed and makes our embedding vertical. Define $\mathscr{G}$ to be the composition of these ambient isotopies. This completes the first step.

The *inductive step* k to $k+1$: By hypothesis there is a neighbourhood U_k of $A_k = B_1\cup\cdots\cup B_k$ in M and an ambient isotopy $\mathscr{H}^k$ of $E\times I$, which keeps $E\times O$ fixed, such that $\mathscr{H}^k_1 F$ is vertical over U_k. Let B be a locally flat ball in M which contains B_{k+1} in its interior, regard B as a piecewise linear manifold, and let α be an ambient isotopy of $E\times O$ such that $\alpha_1 f \mid B$ is a piecewise linear embedding. Apply $\alpha\times 1$ to $E\times I$ and let G denote the embedding

$$(\alpha\times 1)\,\mathscr{H}^k_1 F: M\times I\to E\times I\,.$$

Note that G is vertical over U_k and $G_0 \mid B$ is piecewise linear.

Now $G(A_k\times I)$ and $G[(B-\operatorname{int} U_k)\times I]$ are disjoint compact subsets of $E\times I$. Therefore there exists a neighbourhood O of $G(A_k\times I)$ in $E\times I$ disjoint from $G[(B-\operatorname{int} U_k)\times I]$. Let V be an open neighbourhood of $G_0 A_k$ in $E\times O$ such that $V\times I\subset 0$. Triangulate $E\times O$ so that $G_0 B$ is a subcomplex and subdivide finely enough so that the simplicial neighbourhood S of $G_0 A_k$ lies entirely in V. Let N be a second derived neighbourhood of S that lies in V. Then N is a compact piecewise linear q-submanifold of $E\times O$ such that

$$G(A_k\times I)\subset \operatorname{int} N\times I\,,$$

and

$$(N\times I)\cap G(B\times I)\subset G(U_k\times I)\,.$$

Let Q denote the piecewise linear q-manifold $E\times O-\operatorname{int} N$. Let $W=(B\times O)\cap G_0^{-1}Q$ and $P=(B\times O)\cap G_0^{-1}\partial Q$; then W is a compact top dimensional piecewise linear submanifold of $B\times O$ and P a compact top dimensional piecewise linear submanifold of ∂W. Also $G \mid W\times I$ is a concordance of W in Q; it is vertical over P and piecewise linear on $(W\times O)\cup(P\times I)$. Suppose there is an ambient isotopy $\mathscr{G}$ of $Q\times I$, fixed on $(Q\times O)\cup(\partial Q\times I)$, such that $\mathscr{G}_1 G \mid W\times I$ is vertical. Extend $\mathscr{G}$ by the identity to an ambient isotopy of the whole of $E\times I$. Then $\mathscr{H}^{k+1}$ can be defined as the following composition: $\mathscr{H}^k$ followed by $\alpha\times 1$ followed by $\mathscr{G}$ followed by the inverse of $\alpha\times 1$. Note that $\mathscr{H}^{k+1}$ keeps $E\times O$ fixed and that $\mathscr{H}^{k+1}_1 F$ is vertical over the neighbourhood

$$U_{k+1}=(N\cap M)\cup B$$

of A_{k+1} in M.

It remains to define $\mathcal{G}$. Again $\mathcal{G}$ will be a composition. A couple of applications of Theorem 3 and Miller's theorem (first in $Q \times 1$, then in $Q \times I$) provide an ambient isotopy of $Q \times I$, fixed on $(Q \times O) \cup (\partial Q \times I)$, that makes $G \mid W \times I$ a piecewise linear concordance. Applying Hudson's theorem, this new embedding can be made vertical by an ambient isotopy of $Q \times I$ that holds $(Q \times O) \cup (\partial Q \times I)$ fixed. Define $\mathcal{G}$ to be the composition of these two ambient isotopies.

Theorem 4 only relates to points if $q \leqslant 3$. When $q = 4$ the ends of the given concordance can be ambient isotoped piecewise linear since M is at worst a finite number of lines and circles. Theorem 3 and Miller's theorem make the concordance completely piecewise linear, and so Hudson's result can be used to straighten it. This completes Theorem 4.

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