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# Operational Calculus for Two Commuting Closed Operators 

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## 1. Introduction

Functions of an endomorphism $T$ of a complex Banach space $\mathfrak{X}$ can be defined through a proper homomorphic mapping $\Phi$ of an algebra $H$ into the Banach algebra $B(\mathfrak{X})$ of all endomorphism of $\mathfrak{X}$. The elements of $H$ are locally holomorphic complex functions $f(\lambda)$ defined on a neighborhood of the spectrum $\sigma(T)$ of $T$. If $\Phi$ is continuous and the image of $\lambda$ under $\Phi$ is $T$, then $\Phi$ is unique and is represented by a Cauchy integral: $\Phi(f)=(2 \pi i)^{-1} \int_{\Gamma} f(\lambda)(\lambda-T)^{-1} d \lambda$, where $\Gamma$ is a suitably chosen curve in the complex plane. $\Phi(f)$ is a function of $T$, and the construction of functions of $T$ by this procedure is known as operational calculus for bounded linear operators. The Cauchy formula has been used by Dunford [2, 3, 4] and by Taylor [10] to define functions of an endomorphism of $\mathfrak{X}$. Such a function could be defined also by a power series expansion in $T$, valid in a neighborhood of $\sigma(T)$, and the result is equivalent to the result obtainable with Cauchy's formula. However, if $T$ is unbounded and closed, a generalization is possible only by use of the integral formula (TAYLOR [11], Hille and Phillips [7]). Another extension of the theory is the generalization to functions of more than one variable, which goes straightforward only in the case of commuting endomorphisms of $\mathfrak{X}$. Such functions have been introduced by Schwartz [8] and Waelbroek [13], where uniqueness is proven in the last cited work.

The theory presented in this note deals with functions of two (not necessarily bounded) commuting closed linear operators $T_{1}$ and $T_{2}$ on $\mathfrak{X}$ to itself. Again, the functions are defined through a proper homomorphism $\Phi$ of an algebra $H$ into $B(\mathfrak{X})$. Here $H$ consists of locally holomorphic complex functions $f\left(\lambda_{1}, \lambda_{2}\right)$ whose domain $V$ is a neighborhood of $\sigma_{e}\left(T_{1}\right) \times \sigma_{e}\left(T_{2}\right), \sigma_{e}\left(T_{j}\right)$ being the extended spectrum of $T_{j}$. This implies that $V$ is unbounded, if $T_{1}$ or $T_{2}$ is unbounded. In principle, the main theorem describes the following facts: Provided that $\Phi$ transforms $\left(\lambda-\lambda_{j}\right)^{-1}$ into $\left(\lambda-T_{j}\right)^{-1}$ for $j=1,2$ and for each $\lambda$ in the resolvent set of $T_{j}, \Phi$ is unique and has the representation given by formula (1).

The operational calculus for closed linear operators plays an important role in the theory of Banach algebras [12, 14], in the spectral theory [4], the theory of semigroups [7] and in quantum theory [1]. As an application we shall give a representation for polynomials in two endomorphisms of $\mathfrak{X}$. Furthermore, we derive formulas for the resolvents of the sum and the product of commuting closed linear transformations and show that the extended spectrum of the sum (product) is contained in the sum (product) of the extended spectra of the single transformations.

## 2. The operational calculus

Let $V_{1}$ and $V_{2}$ be Cauchy domains, open subsets of the extended complex plane $\mathbf{C}$ with a finite positive number of components whose boundaries are nonempty and are composed of a finite positive number of simple closed rectifiable curves. Let $V=V_{1} \times V_{2}$ be the Cartesian product of $V_{1}$ and $V_{2}$ in $\mathbf{C}^{2}$. We denote by $\chi_{X}$ the characteristic function of a set $X$. Let $H(V)$ then be the complex algebra of all functions $f: \mathbf{C}^{2} \rightarrow \mathbf{C}$ which are locally holomorphic in $V$ and vanish in the complement of $V$ with respect to $\mathbf{C}^{2}$. In $H(V)$ we use the ordinary definition of arithmetic operations. $\chi_{V}$ is the unit element of $H(V)$.

Now let $U$ be a Cauchy domain and $X$ a closed subset of $U$. Then there exists a Cauchy domain $U^{\prime}$, such that $X \subset U^{\prime}$ and $\overline{U^{\prime}} \subset U[11$, Theorem 3.3]. The union $\Gamma$ of all boundaries of $U^{\prime}$, with the usual positive orientation, we call an oriented envelope of $X$ with respect to $U$.

Next, let $\mathfrak{X}$ be a complex Banach space, $\mathfrak{D}(\mathfrak{X})$ the class of all closed linear transformations with domain and range in $\mathfrak{X}$ and $B(\mathfrak{X})$ the complex Banach algebra of all endomorphisms of $\mathfrak{X}$ with identity $I$. Let $T$ be an element of $\mathfrak{D}(\mathfrak{X})$ with domain $\mathfrak{D}(T)$, spectrum $\sigma(T)$ and resolvent set $\sigma(T)$. The extended spectrum $\sigma_{e}(T)$ is the set of all singular points of the resolvent $R(\lambda, T)$ of $T$ in $\mathbf{C}$. By $\mathbb{C}(\mathfrak{X})$ we denote the class of all pairs $\left\{T_{1}, T_{2}\right\}$ of operators of $\mathfrak{D}(\mathfrak{X})$ which satisfy $\mathfrak{D}\left(T_{1}\right) \subset \mathfrak{D}\left(T_{2}\right), T_{2}\left[\mathfrak{D}\left(T_{1}\right)\right] \subset$ $\mathfrak{D}\left(T_{1}\right)$ and commute on the set $\left\{x \mid x \in \mathfrak{D}\left(T_{1}\right), T_{1} x \in \mathfrak{D}\left(T_{2}\right)\right\}$. If $T_{2}$ is bounded on $\mathfrak{X}$, this definition is equivalent to that of M. H. Stone [9, Definition 3]. Then we have

Lemma 1. Let $\mathfrak{D}\left(T_{1}\right) \subset \mathfrak{D}\left(T_{2}\right)$ and $T_{2}\left[\mathfrak{D}\left(T_{1}\right)\right] \subset \mathfrak{D}\left(T_{1}\right)$. A necessary and sufficient condition that $R\left(\lambda_{1}, T_{1}\right) R\left(\lambda_{2}, T_{2}\right)=R\left(\lambda_{2}, T_{2}\right) R\left(\lambda_{1}, T_{1}\right)$ for each pair $\left\{\lambda_{1}, \lambda_{2}\right\} \in \varrho\left(T_{1}\right)$ $\times \varrho\left(T_{2}\right)$ is that $\left\{T_{1}, T_{2}\right\} \in \mathbb{C}(\mathfrak{X})$.

Proof. If $\left\{T_{1}, T_{2}\right\} \in \mathfrak{C}(\mathfrak{X})$ we have $\left\{x \mid x \in \mathfrak{D}\left(T_{1}\right), T_{1} x \in \mathfrak{D}\left(T_{2}\right)\right\}=\left\{x \mid x \in \mathfrak{D}\left(T_{1}\right)\right.$, $\left.\left(\lambda_{1} I-T_{1}\right) x \in \mathfrak{D}\left(T_{2}\right)\right\}=R\left(\lambda_{1}, T_{1}\right)\left[\mathfrak{D}\left(T_{2}\right)\right]=R\left(\lambda_{1}, T_{1}\right) R\left(\lambda_{2}, T_{2}\right)(\mathfrak{X})$, since the $R\left(\lambda_{j}, T_{j}\right)$ are one-to-one transformations of $\mathfrak{X}$ onto $\mathfrak{D}\left(T_{j}\right)$. For each $x \in \mathfrak{X}$

$$
\begin{aligned}
R\left(\lambda_{2}, T_{2}\right) R\left(\lambda_{1}, T_{1}\right)\left[( \lambda _ { 1 } I - T _ { 1 } ) \left(\lambda_{2}\right.\right. & \left.\left.I-T_{2}\right)-\left(\lambda_{2} I-T_{2}\right)\left(\lambda_{1} I-T_{1}\right)\right] \\
& \times R\left(\lambda_{1}, T_{1}\right) R\left(\lambda_{2}, T_{2}\right) x \\
& =\left[R\left(\lambda_{1}, T_{1}\right) R\left(\lambda_{2}, T_{2}\right)-R\left(\lambda_{2}, T_{2}\right) R\left(\lambda_{1}, T_{1}\right)\right] x
\end{aligned}
$$

Hence a sufficient condition for the right-hand side to vanish is that $T_{1}$ and $T_{2}$ commute, i.e. $\left\{T_{1}, T_{2}\right\} \in \mathcal{C}(\mathfrak{X})$. On the other hand, if the right-hand side is zero, again since the resolvents are one-to-one, we have $\left(T_{1} T_{2}-T_{2} T_{1}\right) R\left(\lambda_{1}, T_{1}\right) R\left(\lambda_{2}, T_{2}\right) x=0$. Thus $\left\{T_{1}, T_{2}\right\} \in \mathscr{C}(\mathfrak{X})$ and this proves the necessary condition, q.e.d.

Finally, we denote by $\mathfrak{G}(\mathfrak{X}, V)$ the class of all elements of $\mathfrak{C}(\mathfrak{X})$ for which $\sigma_{e}\left(T_{1}\right) \times \sigma_{e}\left(T_{2}\right) \subset V$. Based on the preceding definitions we have

Theorem 2. For each pair $\left\{T_{1}, T_{2}\right\} \in \mathfrak{G}(\mathfrak{X}, V)$ there exists a proper homomorphism $\Phi$ of $H(V)$ into $B(\mathfrak{X})$ such that
(i) $\Phi\left[\left(\lambda-\lambda_{j}\right)^{-1} \chi_{V}\left(\lambda_{1}, \lambda_{2}\right)\right]=R\left(\lambda, T_{j}\right), \quad \lambda \notin V_{j}, \quad j=1,2$.
(ii) If a sequence $f_{n}$ in $H(V)$ converges pointwise to $f \in H(V)$, the convergence being uniform on each compact subset of $V$, then this implies $\lim _{n}\left\|\Phi\left(f_{n}\right)-\Phi(f)\right\|=0$.
(iii) $\Phi$ is unique and is defined by

$$
\begin{align*}
\Phi(f) & =f(\infty, \infty) I+\frac{1}{2 \pi i} \int_{\Gamma_{1}} f\left(\lambda_{1}, \infty\right) R\left(\lambda_{1}, T_{1}\right) d \lambda_{1} \\
& +\frac{1}{2 \pi i} \int_{\Gamma_{2}} f\left(\infty, \lambda_{2}\right) R\left(\lambda_{2}, T_{2}\right) d \lambda_{2}  \tag{1}\\
& +\frac{1}{(2 \pi i)^{2}} \int_{\Gamma_{1}} \int_{\Gamma_{2}} f\left(\lambda_{1}, \lambda_{2}\right) R\left(\lambda_{1}, T_{1}\right) R\left(\lambda_{2}, T_{2}\right) d \lambda_{1} d \lambda_{2},
\end{align*}
$$

where $\Gamma_{j}, j=1,2$ is an oriented envelope of $\sigma_{e}\left(T_{j}\right)$ with respect to $V_{j}$, containing $\lambda=\infty$ in its interior if $V_{j}$ is unbounded.

Proof. Since the integrands of (1) are locally holomorphic in $V \cap\left[\varrho\left(T_{1}\right) \times \varrho\left(T_{2}\right)\right]$ the integrals exist and are independent of the choice of $\Gamma_{1}$ and $\Gamma_{2}$. Clearly $\Phi$, defined by (1), is linear. In order to demonstrate that it is a homomorphism we take $f, g \in H(V)$ and for $j=1,2$ two oriented envelopes $\Gamma_{j}$ and $\Gamma_{j}^{\prime}$ of $\sigma_{e}\left(T_{j}\right)$ with respect to $V_{j}$ such that the open set bounded by $\Gamma_{j}$ contains $\Gamma_{j}^{\prime}$. Then by (1) with $c=(2 \pi i)^{-1}$,

$$
\begin{equation*}
\Phi(f) \Phi(g)-\Phi(f g)=\sum_{k=1}^{8} A_{k} \tag{2}
\end{equation*}
$$

where

$$
\begin{aligned}
A_{1}= & c \int_{\Gamma_{1}}\left[f\left(\lambda_{1}, \infty\right) g(\infty, \infty)+f(\infty, \infty) g\left(\lambda_{1}, \infty\right)-(f g)\left(\lambda_{1}, \infty\right)\right] R\left(\lambda_{1}, T_{1}\right) d \lambda_{1} \\
A_{2}= & c \int_{\Gamma_{2}}\left[f\left(\infty, \lambda_{2}\right) g(\infty, \infty)+f(\infty, \infty) g\left(\infty, \lambda_{2}\right)-(f g)\left(\infty, \lambda_{2}\right)\right] R\left(\lambda_{2}, T_{2}\right) d \lambda_{2} \\
A_{3}= & c^{2} \int_{\Gamma_{1}} \int_{\Gamma_{1}{ }^{\prime}} f\left(\lambda_{1}, \infty\right) g\left(\mu_{1}, \infty\right) R\left(\lambda_{1}, T_{1}\right) R\left(\mu_{1}, T_{1}\right) d \lambda_{1} d \mu_{1} \\
A_{4}= & c^{2} \int_{\Gamma_{2}} \int_{\Gamma_{2}} f\left(\infty, \lambda_{2}\right) g\left(\infty, \mu_{2}\right) R\left(\lambda_{2}, T_{2}\right) R\left(\mu_{2}, T_{2}\right) d \lambda_{2} d \mu_{2} \\
A_{5}= & c^{2} \int_{\Gamma_{1}} \int_{\Gamma_{2}}\left[f\left(\infty, \lambda_{2}\right) g\left(\lambda_{1}, \infty\right)+f\left(\lambda_{1}, \infty\right) g\left(\infty, \lambda_{2}\right)+f(\infty, \infty) g\left(\lambda_{1}, \lambda_{2}\right)\right. \\
& \left.+f\left(\lambda_{1}, \lambda_{2}\right) g(\infty, \infty)-(f g)\left(\lambda_{1}, \lambda_{2}\right)\right] R\left(\lambda_{1}, T_{1}\right) R\left(\lambda_{2}, T_{2}\right) d \lambda_{1} d \lambda_{2}
\end{aligned}
$$

$$
\begin{aligned}
A_{6}= & c^{3} \iint_{\Gamma_{1}} \int_{\Gamma_{1}^{\prime}}\left[f\left(\mu_{\Gamma_{2}}, \infty\right) g\left(\lambda_{1}, \lambda_{2}\right)+f\left(\lambda_{1}, \lambda_{2}\right) g\left(\mu_{1}, \infty\right)\right] \\
& \times R\left(\lambda_{1}, T_{1}\right) R\left(\mu_{1}, T_{1}\right) R\left(\lambda_{2}, T_{2}\right) d \lambda_{1} d \mu_{1} d \lambda_{2} \\
A_{7}= & c^{3} \iint_{\Gamma_{1}} \int_{\Gamma_{2}} \int_{\Gamma_{2^{\prime}}}\left[f\left(\infty, \mu_{2}\right) g\left(\lambda_{1}, \lambda_{2}\right)+f\left(\lambda_{1}, \lambda_{2}\right) g\left(\infty, \mu_{2}\right)\right] \\
& \times R\left(\lambda_{1}, T_{1}\right) R\left(\lambda_{2}, T_{2}\right) R\left(\mu_{2}, T_{2}\right) d \lambda_{1} d \lambda_{2} d \mu_{2} \\
A_{8}= & c^{4} \iint_{\Gamma_{1}} \int_{\Gamma_{2}} \int_{\Gamma_{1}^{\prime}} \int_{\Gamma_{2}^{\prime}} f\left(\lambda_{1}, \lambda_{2}\right) g\left(\mu_{1}, \mu_{2}\right) R\left(\lambda_{1}, T_{1}\right) R\left(\lambda_{2}, T_{2}\right) R\left(\mu_{1}, T_{1}\right) \\
& \times R\left(\mu_{2}, T_{2}\right) d \lambda_{1} d \lambda_{2} d \mu_{1} d \mu_{2} .
\end{aligned}
$$

In some terms we have changed the order of integration and used the commutativity of the resolvents (Lemma 1). By [11, p. 196] we have for $j=1,2$ and any $h \in H(V)$ as a result of the method of residues
and

$$
c \int_{\Gamma_{j}} \frac{h\left(\mu_{j}, \cdot\right)}{\mu_{j}-\lambda_{j}} d \mu_{j}=h\left(\lambda_{j}, \cdot\right)-h(\infty, \cdot), \quad \lambda_{j} \in \Gamma_{j}^{\prime}
$$

$$
c \int_{\Gamma_{j}^{\prime}} \frac{h\left(\mu_{j}, \cdot\right)}{\mu_{j}-\lambda_{j}} d \mu_{j}=-h(\infty, \cdot), \quad \lambda_{j} \in \Gamma_{j}
$$

Using this and the resolvent equation $\left(\mu_{1}-\lambda_{1}\right) R\left(\lambda_{1}, T_{1}\right) R\left(\mu_{1}, T_{1}\right)=R\left(\lambda_{1}, T_{1}\right)-$ $R\left(\mu_{1}, T_{1}\right)$ we get

$$
\begin{aligned}
A_{3}= & c^{2} \int_{\Gamma_{1}} R\left(\lambda_{1}, T_{1}\right) f\left(\lambda_{1}, \infty\right) d \lambda_{1} \int_{\Gamma_{1}^{\prime}} \frac{g\left(\mu_{1}, \infty\right)}{\mu_{1}-\lambda_{1}} d \mu_{1} \\
& +c^{2} \int_{\Gamma_{1}^{\prime}} R\left(\mu_{1}, T_{1}\right) g\left(\mu_{1}, \infty\right) d \mu_{1} \int_{\Gamma_{1}} \frac{f\left(\lambda_{1}, \infty\right)}{\lambda_{1}-\mu_{1}} d \lambda_{1} \\
= & -A_{1}
\end{aligned}
$$

and similarly $A_{4}=-A_{2}$. In the same manner we obtain

$$
\begin{aligned}
A_{8}= & c^{3} \iint_{\Gamma_{1}} \int_{\Gamma_{1}^{\prime}} \int_{\Gamma_{2}}\left[f\left(\lambda_{1}, \lambda_{2}\right) g\left(\mu_{1}, \lambda_{2}\right)-f\left(\lambda_{1}, \infty\right) g\left(\mu_{1}, \lambda_{2}\right)-f\left(\lambda_{1}, \lambda_{2}\right) g\left(\mu_{1}, \infty\right)\right] \\
& \times R\left(\lambda_{1}, T_{1}\right) R\left(\mu_{1}, T_{1}\right) R\left(\lambda_{2}, T_{2}\right) d \lambda_{1} d \mu_{1} d \lambda_{2},
\end{aligned}
$$

hence

$$
\begin{aligned}
A_{6}+A_{8}= & c^{3} \iint_{\Gamma_{1}} \int_{\Gamma_{1}^{\prime}} f\left(\lambda_{1}, \lambda_{2}\right) g\left(\mu_{1}, \lambda_{2}\right) R\left(\lambda_{1}, T_{1}\right) R\left(\mu_{1}, T_{1}\right) R\left(\lambda_{2}, T_{2}\right) d \lambda_{1} d \mu_{1} d \lambda_{2} \\
= & c^{2} \int_{\Gamma_{1}} \int_{\Gamma_{2}}\left[(f g)\left(\lambda_{1}, \lambda_{2}\right)-f\left(\infty, \lambda_{2}\right) g\left(\lambda_{1}, \lambda_{2}\right)-f\left(\lambda_{1}, \lambda_{2}\right) g\left(\infty, \lambda_{2}\right)\right] \\
& \times R\left(\lambda_{1}, T_{1}\right) R\left(\lambda_{2}, T_{2}\right) d \lambda_{1} d \lambda_{2} .
\end{aligned}
$$

Since analogously

$$
\begin{aligned}
A_{7}= & c^{2} \iint_{\Gamma_{1}}\left[-f\left(\infty, \lambda_{2}\right) g\left(\lambda_{1}, \infty\right)-f\left(\lambda_{1}, \infty\right) g\left(\infty, \lambda_{2}\right)-f(\infty, \infty)\right. \\
& \times g\left(\lambda_{1}, \lambda_{2}\right)-f\left(\lambda_{1}, \lambda_{2}\right) g(\infty, \infty) \\
& \left.+f\left(\infty, \lambda_{2}\right) g\left(\lambda_{1}, \lambda_{2}\right)+f\left(\lambda_{1}, \lambda_{2}\right) g\left(\infty, \lambda_{2}\right)\right] R\left(\lambda_{1}, T_{1}\right) R\left(\lambda_{2}, T_{2}\right) d \lambda_{1} d \lambda_{2},
\end{aligned}
$$

we have $A_{7}+A_{5}=-A_{6}-A_{8}$, so that the right-hand side of (2) vanishes and $\Phi(f g)=\Phi(f) \Phi(g)$.

Next, to prove (i) we write down two well-known relations from the operational calculus in one variable [7, Theorem 5.11.2]

$$
\begin{equation*}
c \int_{\Gamma_{j}} R\left(\lambda_{j}, T_{j}\right) d \lambda_{j}=\left[1-\chi_{V_{j}}(\infty)\right] I \tag{3}
\end{equation*}
$$

and [11, Theorem 7.4] (after deforming the path of integration)

$$
\begin{equation*}
c \int_{r_{j}}\left(\lambda-\lambda_{j}\right)^{-1} R\left(\lambda_{j}, T_{j}\right) d \lambda_{j}=R\left(\lambda, T_{j}\right), \quad \lambda \notin V_{j} \tag{4}
\end{equation*}
$$

Using equations (1), (3) and (4) we get

$$
\Phi\left[\left(\lambda-\lambda_{1}\right)^{-1} \chi_{V}\left(\lambda_{1}, \lambda_{2}\right)\right]=\chi_{V_{2}(\infty)} R\left(\lambda, T_{1}\right)+\left[1-\chi_{V_{2}(\infty)}\right] R\left(\lambda, T_{1}\right)=R\left(\lambda, T_{1}\right)
$$

and

$$
\Phi\left[\left(\lambda-\lambda_{2}\right)^{-1} \chi_{V}\left(\lambda_{1}, \lambda_{2}\right)\right]=\chi_{V_{1}(\infty)} R\left(\lambda, T_{2}\right)+\left[1-\chi_{V_{1}(\infty)}\right] R\left(\lambda, T_{2}\right)=R\left(\lambda, T_{2}\right)
$$

From (1) and (3) it is clear that the unit element in the subalgebra $\Phi[H(V)]$ of $B(\mathfrak{X})$ is

$$
\begin{aligned}
\Phi\left(\chi_{V}\right)= & \left\{\chi_{V}^{(\infty, \infty)}+\chi_{V_{2}(\infty)}\left[1-\chi_{V_{1}(\infty)}\right]+\chi_{V_{1}(\infty)}\left[1-\chi_{V_{2}(\infty)}\right]\right. \\
& \left.+\left[1-\chi_{V_{1}(\infty)}\right]\left[1-\chi_{V_{2}(\infty)}\right]\right\} I \\
= & I,
\end{aligned}
$$

therefore $\Phi$ is a proper homomorphism.

Now we take a pair $\left\{\mu_{1}, \mu_{2}\right\}$ of finite points of $\mathbf{C}^{2}-V$. For $g \in H(V)$ we have by (1)

$$
\begin{aligned}
\Phi(g)= & c^{2} \iint_{\Gamma_{1}} g\left(\lambda_{1}, \lambda_{2}\right)\left[\left(\lambda_{1}-\mu_{1}\right)^{-1}\left(\lambda_{2}-\mu_{2}\right)^{-1} I-\left(\lambda_{1}-\mu_{1}\right)^{-1} R\left(\lambda_{2}, T_{2}\right)-\right. \\
& \left.-\left(\lambda_{2}-\mu_{2}\right)^{-1} R\left(\lambda_{1}, T_{1}\right)+R\left(\lambda_{1}, T_{1}\right) R\left(\lambda_{2}, T_{2}\right)\right] d \lambda_{1} d \lambda_{2} .
\end{aligned}
$$

Let $f_{n} \in H(V)$ converge uniformly to $f \in H(V)$ on $\Gamma_{1} \times \Gamma_{2}$. Then $\lim _{n}\left\|\Phi\left(f_{n}\right)-\Phi(f)\right\|=0$, since $\Gamma_{1} \times \Gamma_{2}$ is compact and the expression in the square brackets is bounded on $\Gamma_{1} \times \Gamma_{2}$. This proves the continuity property (ii).

In order to prove the uniqueness of (1) we take a homomorphism of $H(V)$ into $B(\mathfrak{X})$ which satisfies properties (i) and (ii). Then we show that $\Phi$ has the desired representation (1). Due to (ii) we have in the uniform operator topology $\Phi(f)=\lim _{n} \Phi\left(f_{n}\right)$, if there exists a sequence $f_{n}$ in $H(V)$ converging pointwise and uniformly to $f \in H(V)$ on each compact subset $X$ of $V$. But by Cauchy's formula we have on $X=X_{1} \times X_{2}$ :

$$
\begin{aligned}
f\left(\lambda_{1}, \lambda_{2}\right)= & f(\infty, \infty)+c \int_{\Gamma_{1^{\prime}}} \frac{f\left(\mu_{1}, \infty\right)}{\mu_{1}-\lambda_{1}} d \mu_{1}+c \int_{\Gamma_{2^{\prime}}} \frac{f\left(\infty, \mu_{2}\right)}{\mu_{2}-\lambda_{2}} d \mu_{2} \\
& +c^{2} \int_{\Gamma_{1}^{\prime}} \int_{\Gamma_{2}^{\prime}} \frac{f\left(\mu_{1}, \mu_{2}\right)}{\left(\mu_{1}-\lambda_{1}\right)\left(\mu_{2}-\lambda_{2}\right)} d \mu_{1} d \mu_{2},
\end{aligned}
$$

where $\Gamma_{j}^{\prime}$ are oriented envelopes of $X_{j} \cup \sigma_{e}\left(T_{j}\right)$ with respect to $V_{j}$ containing $\lambda=\infty$ in its interior if $V_{j}$ is unbounded. By $V_{j}^{\prime}$ we denote the Cauchy domain enclosed by $\Gamma_{j}^{\prime}$. Since $X_{j} \cup \sigma_{e}\left(T_{j}\right)$ is closed and the boundary $\Gamma_{j}$ of $V_{j}$ is compact, the distance $\delta_{j}$ between $X_{j} \cup \sigma_{e}\left(T_{j}\right)$ and $\Gamma_{j}$ is positive. $\Gamma_{j}^{\prime}$ may be chosen such that the greatest distance between $\Gamma_{j}$ and the points of $\Gamma_{j}^{\prime}$ is less than $\frac{1}{2} \min \left\{\delta_{j}, 1 / M_{j}\right\}$, where $M_{j}=\sup \left\{\left\|R\left(\lambda_{j}, T_{j}\right)\right\| \mid \lambda_{j} \in \bar{V}_{j}-V_{j}\right\}$. Hence there exist suitably chosen sequences of points $\mu_{j 1}, \ldots, \mu_{j n}=\mu_{j o}$ on $\Gamma_{j}^{\prime}$ and $v_{j 1}, \ldots, v_{j n}=v_{j o}$ on $\Gamma_{j}$, such that $\left|\mu_{j k}-\mu_{j, k-1}\right| \rightarrow 0$ uniformly with respect to $k$ for $n \rightarrow \infty$ and $\left|v_{j k}-\mu_{j k}\right|<\frac{1}{2} \min \left\{\delta_{j}, 1 / M_{j}\right\}$ for all $k$. To simplify notation the dependence on $n$ of the points $\mu_{j k}$ and $v_{j k}$ is not indicated. Since the integrals are limits of Riemann sums we have for each $\varepsilon>0$ an $m$ such that on $X$ for $n \geqslant m$

$$
\begin{aligned}
& \left\lvert\, f\left(\lambda_{1}, \lambda_{2}\right)-f(\infty, \infty)-c \sum_{k=1}^{n} \frac{\mu_{1 k}-\mu_{1, k-1}}{\mu_{1 k}-\lambda_{1}} f\left(\mu_{1 k}, \infty\right)\right. \\
& -c \sum_{l=1}^{n} \frac{\mu_{2 l}-\mu_{2, l-1}}{\mu_{2 l}-\lambda_{2}} f\left(\infty, \mu_{2 l}\right) \\
& \left.-c^{2} \sum_{k, l=1}^{n} \frac{\left(\mu_{1 k}-\mu_{1, k-1}\right)\left(\mu_{2 l}-\mu_{2, l-1}\right)}{\left(\mu_{1 k}-\lambda_{1}\right)\left(\mu_{2 l}-\lambda_{2}\right)} f\left(\mu_{1 k}, \mu_{2 l}\right) \right\rvert\,<\varepsilon .
\end{aligned}
$$

Unfortunately, this approximation for $f$ does not belong to $H(V)$, so that we have to move the singularities out of $V$. But $\left|\left(v_{j k}-\mu_{j k}\right) /\left(v_{j \boldsymbol{k}}-\lambda_{j}\right)\right|<\frac{1}{2}$ on $X_{j}$ for each $k$. Hence there is an $m_{j}$ such that on $X_{j}$ for $n_{j} \geqslant m_{j}$

$$
\left|\left(\mu_{j k}-\lambda_{j}\right)^{-1}-\sum_{i=0}^{n_{j}} \frac{\left(v_{j k}-\mu_{j k}\right)^{i}}{\left(v_{j k}-\lambda_{j}\right)^{i+1}}\right|<\frac{2 \pi \varepsilon}{L_{j} N},
$$

where $L_{j}$ is the length of $\Gamma_{j}^{\prime}$ and $N=\sup \left\{\left|f\left(\lambda_{1}, \lambda_{2}\right)\right| \lambda_{1} \in \Gamma_{j}^{\prime}, \lambda_{2} \in \Gamma_{2}^{\prime}\right\}$. Thus

$$
\left|f\left(\lambda_{1}, \lambda_{2}\right)-f_{n}\left(\lambda_{1}, \lambda_{2}\right)\right|<3 \varepsilon+\varepsilon^{2} / N
$$

on $X$, where $n \geqslant \max \left\{m, m_{1}, m_{2}\right\}$ and

$$
\begin{aligned}
f_{n}\left(\lambda_{1}, \lambda_{2}\right)= & \chi_{V}\left(\lambda_{1}, \lambda_{2}\right)\left\{f(\infty, \infty)+c \sum_{k=1}^{n}\left(\mu_{1 k}-\mu_{1, k-1}\right) f\left(\mu_{1 k}, \infty\right)\right. \\
& \times \sum_{i=0}^{n} \frac{\left(v_{1 k}-\mu_{1 k}\right)^{i}}{\left(v_{1 k}-\lambda_{1}\right)^{i+1}}+c \sum_{l=1}^{n}\left(\mu_{2 l}-\mu_{2, l-1}\right) f\left(\infty, \mu_{2 l}\right) \sum_{i=0}^{n} \frac{\left(v_{2 l}-\mu_{2 l}\right)^{i}}{\left(v_{2 l}-\lambda_{2}\right)^{i+1}} \\
& +c^{2} \sum_{k, l=1}^{n}\left(\mu_{1 k}-\mu_{1, k-1}\right)\left(\mu_{2 l}-\mu_{2, l-1}\right) f\left(\mu_{1 k}, \mu_{2 l}\right) \\
& \left.\times \sum_{i, j=0}^{n} \frac{\left(v_{1 k}-\mu_{1 k} k\right.}{\left(v_{1 k}-\lambda_{1}\right)^{i+1}\left(v_{2 l}-\mu_{2 l}-\lambda_{2}\right)^{j+1}}\right\} .
\end{aligned}
$$

Clearly $f_{n}$ is a sequence with the required property and by (i) we have in the uniform operator topology

$$
\begin{aligned}
\Phi(f)= & f(\infty, \infty) I+\lim _{n}\left\{c \sum_{k=1}^{n}\left(\mu_{1 k}-\mu_{1, k-1}\right) f\left(\mu_{1 k}, \infty\right)\right. \\
& \times \sum_{i=0}^{n}\left(v_{1 k}-\mu_{1 k}\right)^{i} R\left(v_{1 k}, T_{1}\right)^{i+1}+c \sum_{l=1}^{n}\left(\mu_{2 l}-\mu_{2, l-1}\right) f\left(\infty, \mu_{2 l}\right) \\
& \times \sum_{i=0}^{n}\left(v_{2 l}-\mu_{2 l}\right)^{i} R\left(v_{2 l}, T_{2}\right)^{i+1}+c^{2} \sum_{k, l=1}^{n}\left(\mu_{1 k}-\mu_{1, k-1}\right)\left(\mu_{2 l}-\mu_{2, l-1}\right) \\
& \left.\times f\left(\mu_{1 k}, \mu_{2, l}\right) \sum_{i, j=0}^{n}\left(v_{1 k}-\mu_{1 k}\right)^{i}\left(v_{2 l}-\mu_{2 l}\right)^{j} \times R\left(v_{1 k}, T_{1}\right)^{i+1} R\left(\mu_{2 l}, T_{2}\right)^{j+1}\right\} .
\end{aligned}
$$

From the resolvent equation we easily get

$$
\sum_{i=0}^{n}\left(v_{j k}-\mu_{j k}\right)^{i} R\left(v_{j k}, T_{j}\right)^{i+1}=R\left(\mu_{j k}, T_{j}\right)\left\{I-\left[\left(v_{j k}-\mu_{j k}\right) R\left(v_{j k}, T_{j}\right)\right]^{n+1}\right\}
$$

so that, since $\left\|\left(v_{j k}-\mu_{j k}\right) R\left(v_{j k}, T_{j}\right)\right\|<\frac{1}{2}$,

$$
\begin{aligned}
\Phi(f)= & f(\infty, \infty) I+c \int_{\Gamma_{1}^{\prime}} f\left(\lambda_{1}, \infty\right) R\left(\lambda_{1}, T_{1}\right) d \lambda_{1}+c \int_{\Gamma_{2}^{\prime}} f\left(\infty, \lambda_{2}\right) R\left(\lambda_{2}, T_{2}\right) d \lambda_{2} \\
& +c^{2} \int_{\Gamma_{1}^{\prime}} \int_{\Gamma_{2}^{\prime}} f\left(\lambda_{1}, \lambda_{2}\right) R\left(\lambda_{1}, T_{1}\right) R\left(\lambda_{2}, T_{2}\right) d \lambda_{1} d \lambda_{2}
\end{aligned}
$$

and the proof of Theorem 2 is complete.

## 3. Polynomials, resolvents and spectra of sums and products of operators

In view of an application to polynomials in bounded linear operators we have the following

Corollary 3. Let $V_{j}$ be bounded and let $\left\{T_{1}, T_{2}\right\} \in \mathfrak{F}(\mathfrak{X}, V)$. Then for $j=1$, 2 we have $\Phi\left[\lambda_{j} \chi_{V}\left(\lambda_{1}, \lambda_{2}\right)\right]=T_{j}$.

Proof. Since $T_{j}$ is bounded it immediately follows from (1) and (3) that

$$
\begin{aligned}
& \Phi\left[\lambda_{j} \chi_{V}\left(\lambda_{1}, \lambda_{2}\right)\right]=\frac{1}{2 \pi i} \int_{\Gamma_{j}} \lambda_{j} R\left(\lambda_{j}, T_{j}\right) d \lambda_{j} \\
= & \frac{1}{2 \pi i} \int_{\Gamma_{j}}\left[I+T_{j} R\left(\lambda_{j}, T_{j}\right)\right] d \lambda_{j}=T_{j} . \quad \text { q.e.d. }
\end{aligned}
$$

For the sum of $T_{1}$ and $T_{2}$ we obtain
Theorem 4. Let $\left\{T_{1}, T_{2}\right\} \in \mathfrak{F}(\mathfrak{X}, V)$. Then for each $\lambda \notin V_{1}+V_{2}$ we have

$$
\Phi\left[\left(\lambda-\lambda_{1}-\lambda_{2}\right)^{-1} \chi_{V}\left(\lambda_{1}, \lambda_{2}\right)\right]=R\left(\lambda, T_{1}+T_{2}\right)
$$

Proof. Clearly $\left(\lambda-\lambda_{1}-\lambda_{2}\right)^{-1} \chi_{V}\left(\lambda_{1}, \lambda_{2}\right) \in H(V)$ if there is a $\lambda \notin V_{1}+V_{2}$. Applying the method of residues we get by (1)

$$
\begin{equation*}
\Phi\left[\left(\lambda-\lambda_{1}-\lambda_{2}\right)^{-1} \chi_{V}\left(\lambda_{1}, \lambda_{2}\right)\right]=\frac{1}{2 \pi i} \int_{\Gamma_{1}} R\left(\lambda_{1}, T_{1}\right) R\left(\lambda-\lambda_{1}, T_{2}\right) d \lambda_{1} . \tag{5}
\end{equation*}
$$

If $V_{1}$ is unbounded, then, according to our assumptions, $V_{2}$ must be bounded and vice versa. On $\mathfrak{X}$ we then have

$$
\begin{aligned}
\left(\lambda I-T_{1}-T_{2}\right) \int_{\Gamma_{1}} R & \left(\lambda_{1}, T_{1}\right) R\left(\lambda-\lambda_{1}, T_{2}\right) x d \lambda_{1} \\
& =\left[\lambda_{1} I-T_{1}+\left(\lambda-\lambda_{1}\right) I-T_{2}\right] \int_{\Gamma_{1}} R\left(\lambda_{1}, T_{1}\right) R\left(\lambda-\lambda_{1}, T_{2}\right) x d \lambda_{1} \\
& =\int_{\Gamma_{1}} R\left(\lambda-\lambda_{1}, T_{2}\right) x d \lambda_{1}+\int_{\Gamma_{1}} R\left(\lambda_{1}, T_{1}\right) x d \lambda_{1} \\
& =-\chi_{V_{1}}(\infty) \int_{c_{1}} R\left(\lambda-\lambda_{1}, T_{2}\right) x d \lambda_{1}+2 \pi i\left[1-\chi_{V_{1}}(\infty)\right] x \\
& =\chi_{V_{1}}(\infty) \int_{c_{1}} R\left(\mu, T_{2}\right) x d \mu+2 \pi i\left[1-\chi_{V_{1}}(\infty)\right] x \\
& =2 \pi i x,
\end{aligned}
$$

where we have used the fact that $T_{1}$ and $T_{2}$ commute with the integral, $C_{1}$ is a positively oriented circle of sufficiently large radius around the origin, and $\mu$ has been substituted for $\lambda-\lambda_{1}$. Hence the convolution (5) equals $R\left(\lambda, T_{1}+T_{2}\right)$, q.e.d.

Corollary 5. Let $T_{1}$ or $T_{2}$ be bounded and $\left\{T_{1}, T_{2}\right\} \in \mathbb{C}(\mathfrak{X})$. Then

$$
\sigma_{e}\left(T_{1}+T_{2}\right) \subset \sigma_{e}\left(T_{1}\right)+\sigma_{e}\left(T_{2}\right)
$$

Proof. If $\sigma_{e}\left(T_{1}\right)+\sigma_{e}\left(T_{2}\right)=\mathbf{C}$ the statement is trivial. Otherwise we have for each complex $\lambda \notin \sigma_{e}\left(T_{1}\right)+\sigma_{e}\left(T_{2}\right)$ a neighborhood $V$ (in the sense of Section 2) of $\sigma_{e}\left(T_{1}\right) \times \sigma_{e}\left(T_{2}\right)$ such that $\left(\lambda-\lambda_{1}-\lambda_{2}\right)^{-1} \chi_{V}\left(\lambda_{1}, \lambda_{2}\right) \in H(V)$ and the resolvent $R\left(\lambda, T_{1}+T_{2}\right)$ is given by (5). But since $\lambda-\lambda_{1} \notin \sigma_{e}\left(T_{2}\right)$, the integrand in (5) is bounded on $\Gamma_{1}$ and, since $\Gamma_{1}$ is compact, $R\left(\lambda, T_{1}+T_{2}\right)$ is also bounded. Hence $\lambda \notin \sigma_{e}\left(T_{1}+T_{2}\right)$. This implies $\sigma_{e}\left(T_{1}+T_{2}\right) \subset \sigma_{e}\left(T_{1}\right)+\sigma_{e}\left(T_{2}\right)$, q.e.d.

In a manner similar to that used in Theorem 4 we obtain for the product of $T_{1}$ and $T_{2}$ :

Theorem 6. Let $\left\{T_{1}, T_{2}\right\} \in \mathfrak{G}(\mathfrak{X}, V)$. Then for each $\lambda \notin V_{1} \cdot V_{2}$ we have

$$
\begin{align*}
\Phi\left[\left(\lambda-\lambda_{1} \lambda_{2}\right)^{-1}\right. & \left.\chi_{V}\left(\lambda_{1}, \lambda_{2}\right)\right]= \\
& =\frac{1}{2 \pi i} \int_{\Gamma_{1}} R\left(\lambda_{1}, T_{1}\right) R\left(\lambda / \lambda_{1}, T_{2}\right) \frac{d \lambda_{1}}{\lambda_{1}}=R\left(\lambda, T_{1} T_{2}\right) \tag{6}
\end{align*}
$$

The proof of this theorem parallels that of the preceding theorem. Here we have to show that on $\mathfrak{X}$

$$
\begin{aligned}
\left(\lambda I-T_{1}\right. & \left.T_{2}\right) \int_{\Gamma_{1}} R\left(\lambda_{1}, T_{1}\right) R\left(\lambda / \lambda_{1}, T_{2}\right) x \frac{d \lambda_{1}}{\lambda_{1}} \\
= & \int_{\Gamma_{1}}\left[\lambda I-\lambda_{1} T_{2}-\left(\frac{\lambda}{\lambda_{1}} I-T_{2}\right)\left(\lambda_{1} I-T_{1}\right)+\frac{\lambda}{\lambda_{1}}\left(\lambda_{1} I-T_{1}\right)\right] \\
& \times R\left(\lambda_{1}, T_{1}\right) R\left(\lambda / \lambda_{1}, T_{2}\right) x \frac{d \lambda_{1}}{\lambda_{1}} \\
= & \int_{\Gamma_{1}} R\left(\lambda_{1}, T_{1}\right) x d \lambda_{1}-\int_{\Gamma_{1}} \frac{d \lambda_{1}}{\lambda_{1}} x+\lambda \int_{\Gamma_{1}} R\left(\lambda / \lambda_{1}, T_{2}\right) x \frac{d \lambda_{1}}{\lambda_{1}^{2}} \\
= & 2 \pi i\left[1-\chi_{V_{1}}(\infty)\right] x-2 \pi i\left[\chi_{V_{1} \prime^{\prime}}(0)-\chi_{V_{1}}(\infty)\right] x \\
& +\chi_{V_{1}}(0) \lambda \int_{C_{2}} R\left(\lambda / \lambda_{1}, T_{2}\right) x \frac{d \lambda_{1}}{\lambda_{1}^{2}}-\chi_{V_{1}}(\infty) \lambda \int_{c_{1}} R\left(\lambda / \lambda_{1}, T_{2}\right) x \frac{d \lambda_{1}}{\lambda_{1}^{2}} \\
= & 2 \pi i\left[1-\chi_{V_{1}}(0)\right] x+\chi_{V_{1}}(0) \int_{c_{1}} R\left(\mu, T_{2}\right) x d \mu-\chi_{V_{1}}(\infty) \int_{C_{2}} R\left(\mu, T_{2}\right) x d \mu
\end{aligned}
$$

where $\Gamma_{1}$ is chosen such that it does not contain the point $\lambda=0, V_{1}^{\prime}$ is the Cauchy domain enclosed by $\Gamma_{1}, C_{2}$ is a sufficiently small positively oriented circle around the origin, and $\mu$ has been substituted for $\lambda / \lambda_{1}$. If $\chi_{V 1}(0)=1$, then $\lambda=0$ is in $V_{1}, V_{2}$ is bounded and, according to (3), the first integral in the last expression is $2 \pi i x$. If $\chi_{V_{1}}(\infty) \neq 0$, then $\lambda=0$ is not in $V_{2}$ and the last integral vanishes. Hence the whole expression equals $2 \pi i x$ and this shows that $R\left(\lambda, T_{1} T_{2}\right)$ is given by (6).

Corollary 7. Let $\left\{T_{1}, T_{2}\right\} \in \mathbb{C}(\mathfrak{X})$ and let the point $\lambda=0$ not be contained in the spectrum of one operator if the other is unbounded. Then $\sigma_{e}\left(T_{1} T_{2}\right) \subset \sigma_{e}\left(T_{1}\right) \cdot \sigma_{e}\left(T_{2}\right)$.

Again the proof is similar to that for Corollary 5 with a few changes in signs.
Remark. Corollaries 5 and 7 have been proved by Foguel [5, Corollary 1] for the special case of two commuting scalar operators whose spectra are finite point sets in C. More generally, using the fact that there exists a complex commutative Banach algebra containing $I, T_{1}$ and $T_{2}$, Corollaries 5 and 7 for two commuting bounded linear operators $T_{1}$ and $T_{2}$ on $\mathfrak{X}$ follow from the Gelfand theory [6, Satz 6 and §5]. A convolution integral similar to (5) for the resolvent of the sum of two bounded commuting linear operators has been established by Bianchi and Favella [1] in connection with problems in scattering theory.

## REFERENCES

[1] L. Bianchi and L. Favella, A convolution integral for the resolvent of the sum of two commuting operators, Nuovo Cimento, 14 (1964), 1825-1828.
[2] N. Dunford, Spectral theory I. Convergence to projections, Trans. Amer. Math. Soc. 54 (1943), 185-217.
[3] N. Dunford, Spectral theory, Bull. Amer. Math. Soc. 49 (1943), 637-651.
[4] N. Dunford and J. T. Schwartz, Linear operators, New York I (1958), II (1963).
[5] S. R. Foguel, Sums and products of commuting spectral operators, Ark. Mat. 3 (1957), 449-461.
[6] I. M. Gelfand, Normierte Ringe, Math. Sbornik N.S. 9 (51) (1941), 3-24.
[7] E. Hille and R. S. Philitps, Functional analysis and semi-groups, Amer. Math. Soc. Colloquium Publ. 31 (rev. ed.) (1957).
[8] J. Schwartz, Two perturbation formulae, Comm. Pure Appl. Math. 8 (1955), 371-376.
[9] M. H. Stone, On unbounded operators in Hilbert space, J. Indian Math. Soc. 15 (1951), 155-192.
[10] A. E. Taylor, Analysis in complex Banach spaces, Bull. Amer. Math. Soc. 49 (1943), 652-669.
[11] A. E. Taylor, Spectral theory of closed distributive operators, Acta Math. 84 (1950), 189-224.
[12] A. Trampus, A spectral mapping theorem for functions of two commuting linear operators, Proc. Amer. Math. Soc. 14 (1963), 893-895.
[13] L. Waelbroek, Le calcul symbolique dans les algèbres commutatives, J. Math. Pure et Appl. 33 (1954), 147-186.
[14] L. Waelbroek, Théorie des algèbres de Banach et des algèbres localement convexes, Les presses de l'université de Montréal (1965).

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