On a characterization of quasiconformal mappings.

On a characterization of quasiconformal mappings[1]

by Edgar Reich

1. Introduction

Let $M(Q)$ denote the module of the quadrilateral Q (a Jordan domain with four distinguished boundary points and two non-adjacent distinguished sides), A sense-preserving homeomorphism f of a region Ω onto the image region $f(\Omega)$ is said to be a quasiconformal mapping with maximal dilatation $K(\Omega) = K_f(\Omega)$ if

$$\sup_{\bar{Q} \subset \Omega} \frac{M(f(Q))}{M(Q)} = K(\Omega) < \infty . \tag{1.1}$$

($f(Q)$ denotes the quadrilateral with preimage Q.)

Instead of considering the effect of f on quadrilaterals it is natural to study the effect of f on ring domains (doubly connected regions). The modulus $\mu(R)$ of a ring R is defined as $(2\pi)^{-1} \log(r_2/r_1)$, where $\{r_1 < |w| < r_2\}$ is an annulus conformally equivalent to R. It is well known [1] that if (1.1) holds then

$$\frac{\mu(f(R))}{\mu(R)} \leq K(\Omega) , \quad \bar{R} \subset \Omega . \tag{1.2}$$

In fact, the following has been proved by Gehring and Väisälä ([2], Theorem 3):

$$\sup_{\bar{R} \subset \Omega} \frac{\mu(f(R))}{\mu(R)} = K(\Omega) . \tag{1.3}$$

Thus quasiconformal mappings f with maximal dilatation $K(\Omega)$ may be characterized by (1.3) instead of (1.1).

In the proof of (1.3) in [2] essential use is made of the deep and rather difficult "analytic" characterization of quasiconformal maps (See, for instance, [4], Chapter 4). In view of the significance of (1.3) it appears desirable to obtain a more direct and more elementary proof of (1.3), starting with the definition (1.1). It is the object of the present note to provide such a proof.

2. A class of ring domains

Let Q be a quadrilateral with distinguished sides α_1, α_2. A ring R will be said to *link* Q if every closed curve γ, in R, separating the boundary com-

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ponents of R, contains an arc lying in Q which joins α_1 and α_2. (See Fig. 1 where the heavy lines are the boundary conponents of R, and the dashed line represents Q.)

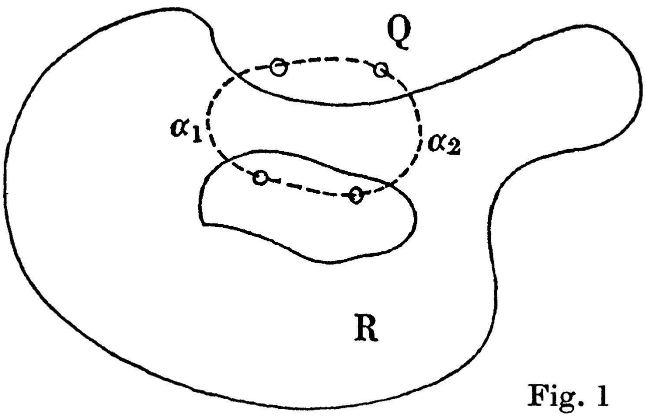

Fig. 1

We will first show that

$$\mu(R) \leq M(Q)\,, \text{ if } R \text{ links } Q\,. \tag{2.1}$$

Let $\varrho^*(z), z \in Q$, be the extremal metric for Q; that is,

$$M(Q) = \frac{A_{\varrho^*}(Q)}{L^2_{\varrho^*}(Q)}\,,\ A_{\varrho^*}(Q) = \iint_Q \varrho^{*2} \mid dz \mid^2,\ L_{\varrho^*}(Q) = \inf_\gamma \int_\gamma \varrho^* \mid dz \mid\,,$$

where $\{\gamma\}$ is the set of locally rectifiable curves in Q joining α_1, α_2. On the other hand, (See, for instance, [3], Chapter 2),

$$\mu(R) = \inf_\varrho \frac{A_\varrho(R)}{L^2_\varrho(R)} \tag{2.2}$$

where, this time, $\{\gamma\}$ is the set of closed curves in R separating the boundary components. For $z \in R$, let

$$\varrho(z) = \begin{cases} \varrho^*(z), z \in R \cap Q \\ \quad 0, z \in R - Q \end{cases}.$$

Clearly, $$A_\varrho(R) \leq A_{\varrho^*}(Q).$$

Since R links Q, $L_\varrho(R) \geq L_{\varrho^*}(Q)$. Hence, by (2.2),

$$\mu(R) \leq \frac{A_{\varrho^*}(Q)}{L^2_{\varrho^*}(Q)} = M(Q)\,,$$

as was to be shown.

In general, to obtain a bound on the modulus of a ring domain R from below, one attempts to make use of the fact ([3], Chapter 2) that

$$\frac{1}{\mu(R)} = \inf_\varrho \frac{A_\varrho(R)}{L^2_\varrho(R)} \tag{2.3}$$

where, now, $\{\gamma\}$ is the set of locally rectifiable curves in R *joining* the boundary components. Fortunately, we shall require a bound on $\mu(R)$ from below, for rings linking Q, only in the very special case considered next.

Suppose the set of positive numbers, a, p, h, s, is given, and

$$2p < \min (a, 1), \; h < a - 2p, \; p < s < a - p - h, \; h < 1 .$$

We consider the ring R' (Fig. 2) obtained as follows.

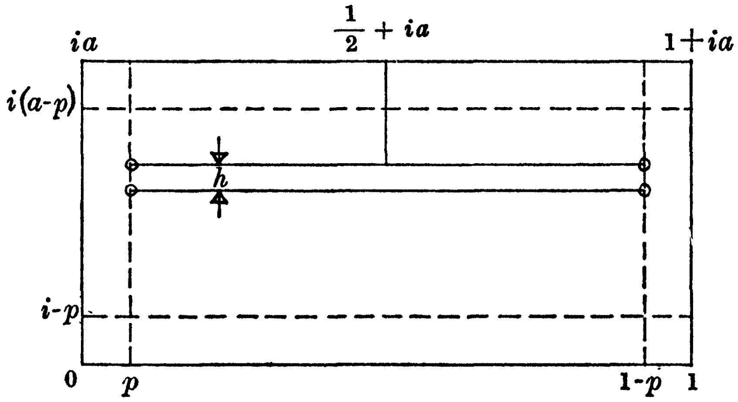

Fig. 2

The outer boundary component of R' consists of the union of (a) the boundary of the rectangle with vertices $0, 1, 1 + ia, ia$ (b) the vertical segment joining the points $\frac{1}{2} + ia, \frac{1}{2} + i(a - s)$ (c) the horizontal segment $I_1 = \{z \mid \Im z = a - s, \; p \leq \Re z \leq 1 - p\}$.

The inner boundary component of R' is the horizontal segment

$$I_2 = \{z \mid \Im z = a - s - h, \; p \leq \Re z \leq 1 - p\} .$$

Let T be the rectangle with horizontal sides I_1, I_2, with the vertical sides (of length h) distinguished. The ring R' links T, and, therefore, by (2.1), $\mu(R') \leq h/(1 - 2p)$. We shall now show that

$$\mu(R') \geq \frac{h}{1 + ah^{\frac{1}{2}}}, \quad \text{if } h \leq p^4 . \tag{2.4}$$

Let

$$\varrho(z) = \begin{cases} 1, & 0 < \Re z < 1, \; a - s - h < \Im z < a - s \\ h^{\frac{3}{4}}, & \text{elsewhere in } R' . \end{cases}$$

Let $\{\gamma\}$ be the family of curves in R' joining I_2 to the outer boundary component of R'. Clearly,

$$L_\varrho(R') = \min (h, h^{\frac{3}{4}} p), \; A_\varrho = h + h^{\frac{3}{2}} (a - h) \leq h + ah^{\frac{3}{2}} .$$

Hence, by (2.3),

$$\frac{1}{\mu(R')} \leq \frac{A_\varrho(R')}{L_\varrho^2(R')} \leq \frac{h + ah^{\frac{3}{2}}}{h^2} = \frac{1 + ah^{\frac{1}{2}}}{h}, \text{ if } h^{\frac{3}{4}} p \geq h .$$

This is equivalent to (2.4).

3. Proof of (1.3)

Let ϵ, $0 < \epsilon < \frac{1}{2}$, be given. In the notation of Section 1, there exists a quadrilateral $Q, \overline{Q} \subset \Omega$, with

$$\frac{M(f(Q))}{M(Q)} \geq (1-\varepsilon)\, K(\Omega)\,.$$

Let the transformation Φ map $f(Q)$ conformally onto a rectangle V with vertices $0, 1, 1+ia, ia$, with the vertical sides distinguished. Thus

$$a = M(V) = M(f(Q))\,.$$

Let V_p be the rectangle, interior to V and oriented as V, such that the distance between corresponding sides of V_p and V is p, $0 < p < \frac{1}{2}\min(a, 1)$ Let $Q_p = f^{-1}\Phi^{-1}(V_p)$. There exists [1] a number $\delta(\epsilon) > 0$ such that

$$\frac{M(V_p)}{M(Q_p)} = \frac{a-2p}{(1-2p)\,M(Q_p)} \geq (1-2\varepsilon)\,K(\Omega)\,, \quad \text{if } p \leq \delta(\varepsilon)\,;$$

that is, Q_p has nearly the same module as Q if p is small.

We now divide V_p into n equal horizontal strips, σ_k, $k = 1, 2, \ldots, n$. σ_k is an $h_n \times 1 - 2p$ rectangle, $h_n = (a-2p)\,n^{-1}$, whose vertical sides are distinguished. The quadrilaterals

$$q_k = f^{-1}\Phi^{-1}(\sigma_k),\ k = 1, 2, \ldots$$

which lie in Ω, subdivide Q_p. According to the subadditivity property for modules [1],

$$\sum_{k=l}^{n} M(q_k) \leq M(Q_p) \leq \frac{a-2p/1-2p}{(1-2\varepsilon)\,K(\Omega)}\,.$$

Hence for some k, say $k = k_n$,

$$M(q_{k_n}) \leq \frac{a-2p/1-2p}{(1-2\varepsilon)\,nK(\Omega)} = \frac{h_n/1-2p}{(1-2\varepsilon)\,K(\Omega)}\,.$$

If we define $N(p)$ as $N(p) = (a-2p)\,p^{-4}$, then $n \geq N(p)$ implies $h_n \leq p^4$.

Let R'_{pn} be the ring linking σ_{k_n} formed as in Fig. 2, with the present h_n serving as the h of Fig. 2. Then, by (2.4),

$$\mu(R'_{pn}) \geq \frac{h_n}{1 + ah_n^{\frac{1}{2}}}\,, \quad \text{if } n \geq N(p)\,.$$

Consider the ring $R_{pn} = f^{-1}\Phi^{-1}(R'_{pn})$, in Ω. Since R_{pn} links q_{k_n}, we have, by (2.1),

$$\mu(R_{pn}) \leq M(q_{kn}) \leq \frac{h_n/1 - 2p}{(1 - 2\varepsilon)K(\Omega)}, \quad \text{if } p \leq \delta(\varepsilon).$$

Now $\mu(f(R_{pn})) = \mu(R'_{pn})$. Therefore,

$$\frac{\mu(f(R_{pn}))}{\mu(R_{pn})} \geq \frac{(1 - 2\varepsilon)(1 - 2p)K(\Omega)}{1 + ah_n^{\frac{1}{2}}}, \quad \text{if } p \leq \delta(\varepsilon), n \geq N(p). \quad (3.1)$$

Once p is chosen, h_n can be made arbitrarily small by taking n sufficiently large. Thus (3.1) shows that there exist rings $R, \bar{R} \subset \Omega$, such that

$$\frac{\mu(f(R))}{\mu(R)} \geq (1 - 3\,\varepsilon)\,K(\Omega).$$

Together with (1.2) the above establishes (1.3).

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