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# On fibre spaces in which the fibre is contractible

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*Dedicated to H. Hopf on his 60<sup>th</sup> birthday*

1. Let  $f : X \rightarrow Y$  be a map of a space  $X$  in a space  $Y$  and let  $A = f^{-1}y_0$ , for some point  $y_0 \in Y$ . Let  $A$  be either a locally finite  $CW$ -complex, as defined in [16], or a compactum which is an  $ANR$ . Then so are  $A \times A$  and<sup>2)</sup>

$$A \vee A = (A \times a_0) \cup (a_0 \times A) \subset A \times A \quad (a_0 \in A) .$$

We assume that the covering homotopy theorem is valid with respect to  $f$  for any map  $g : A \times A \rightarrow X$  and any homotopy of  $f \circ g$ . This will be so if  $f$  determines a fibering of  $X$  with a local product representation ([13], § 11.7, and [7], § 5). Subject to the latter condition we describe  $f$  as a *bundle mapping*. In § 3 below we prove :

**Theorem (1.1).** *If  $A$  is contractible in  $X$  it is an  $H$ -space<sup>3)</sup>.*

The method of proof is suggested by the following observation and an (unpublished) construction, due to M. G. Barratt, for defining the “generalized Whitehead product”,  $[\alpha, \beta] \in \pi_{m+n-1}(X, A)$ , of given elements  $\alpha \in \pi_m(X, A)$ ,  $\beta \in \pi_n(A)$ . Let  $f$  be a fibre mapping. Since  $f_* \beta = 0$  it follows that  $f_*[\alpha, \beta] = [f_* \alpha, 0] = 0$ , where each  $f_*$  denotes the appropriate homomorphism induced by  $f$ . Therefore  $[\alpha, \beta] = 0$ . We have  $\partial[\alpha, \beta] = \pm [\partial\alpha, \beta]$ , where

$$\partial : \pi_{q+1}(X, A) \rightarrow \pi_q(A)$$

is the boundary homomorphism and  $[\beta', \beta] \in \pi_{m+n-2}(A)$  is the ordinary Whitehead product of  $\beta' \in \pi_{m-1}(A)$  and  $\beta$ . Hence it follows that

<sup>1)</sup> This note arose out of consultations during the tenure of a John Simon Guggenheim Memorial Fellowship by Spanier.

<sup>2)</sup> The fact that, if  $A$  is an  $ANR$  compactum, so is  $A \vee A$ , follows from Theorem 1 in [15].

<sup>3)</sup> i. e. there is a map  $h : A \times A \rightarrow A$  such that  $h(a, a_1) = h(a_1, a) = a$  for some  $a_1$  and every  $a \in A$ .

$[\beta', \beta] = 0$  if  $i_*\beta' = 0$ , where  $i_* : \pi_{m-1}(A) \rightarrow \pi_{m-1}(X)$  is the injection. In particular  $\pi_1(A)$  is Abelian if  $X$  is simply connected (cf. [14], p. 289).

Before considering the consequences of (1.1), in its full generality, we draw a corollary from the preceding observation. Let  $X$  be a finite dimensional, locally compact, separable metric space, which is an  $AR$  (absolute retract). Let  $f : X \rightarrow Y$  be a bundle mapping with a connected fibre  $A$ . Since  $X$  is an  $AR$  it follows from an argument on p. 467 of [12] that  $A$  is acyclic and from the above observation that  $\pi_1(A)$  is Abelian. Therefore  $\pi_1(A) \approx H'_1(A) = 0$ , where  $H'_n(A)$  is the  $n^{\text{th}}$  integral, singular homology group of  $A$ . Therefore  $\pi_n(A) \approx H'_n(A) = 0$  for every  $n \geq 1$ . It follows from the local product representation that  $A$  is a neighbourhood retract of  $X$  and hence an  $ANR$ . Since  $X$  is locally compact so, obviously, is  $A$  and since  $\dim A \leq \dim X < \infty$  it follows that  $A$  may be imbedded as a closed<sup>4)</sup> sub-set in some Euclidean space,  $E$ , of which it is a neighbourhood retract. Since  $A$  is connected and  $\pi_n(A) = 0$  for every  $n \geq 1$  it follows that  $A$  is a retract of  $E$ , and hence an  $AR$ . The map  $f$  is obviously open and it follows without difficulty that  $Y$  is a  $C_\sigma$ -space, as defined in § 11.3 of [13]. In particular  $Y$  is covered by a countable set of open sub-sets,  $U_1, U_2, \dots$  such that  $\bar{U}_i$ , the closure of  $U_i$ , lies in a coordinate neighbourhood  $V_i$  (i. e. a neighbourhood such that  $f^{-1}V_i$  is represented as  $A \times V_i$ ). Let  $Y_n = \bar{U}_1 \cup \dots \cup \bar{U}_n$  ( $n \geq 1$ ) and assume that there is a map  $g_n : Y_n \rightarrow X$  such that  $fg_n y = y$  for every  $y \in Y_n$ . It follows from the local product representation that this is so if  $n = 1$ . Let  $T_{n+1} = \bar{U}_{n+1} \cap Y_n$ . Then  $T_{n+1}$  is a closed sub-set of  $\bar{U}_{n+1}$  and the latter is a separable metric space, since it is homeomorphic to a sub-set of  $X$ . Since  $A$  is an  $AR$  it follows that every map  $T_{n+1} \rightarrow A$  has an extension  $\bar{U}_{n+1} \rightarrow A$ . Therefore it follows from the local product representation that  $g_n$  has an extension  $g_{n+1} : Y_{n+1} \rightarrow X$  such that  $fg_{n+1} y = y$  ( $y \in Y_{n+1}$ ). Hence it follows by induction on  $n$  that  $f$  has a right inverse,  $g : Y \rightarrow X$ , by means of which  $Y$  may be imbedded in  $X$  in such a way that  $f$  becomes a retraction. Therefore  $Y$  is an  $AR$  and it follows from § 11.6 in [13] that  $X$ , as a bundle over  $Y$ , is equivalent to the product  $A \times Y$ . That is to say there is a homeomorphism  $h : A \times Y \rightarrow X$ , onto  $X$ , such that  $h(A \times y) = f^{-1}y$  for every  $y \in Y$ . Thus we have :

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<sup>4)</sup> By the addition of a single point,  $c$ , we can imbed  $A$  in a compactum,  $C$ , such that  $\dim C \leq \dim A + 1$ . We imbed  $C$  in a  $p$ -sphere,  $S^p$ , for some large value of  $p$ , and  $E = S^p - c$ .

**Theorem (1.2).** *Any fibre bundle with a connected fibre, which, as a space, is a finite dimensional, locally compact, separable metric  $AR$ , is equivalent to a product bundle.*

It follows from the arguments in [13] that, if  $G$  is a topological transformation group of  $A$  and if  $X$  is a bundle with  $G$  as its group, then (1.2) is valid if equivalence is interpreted as equivalence with respect to  $G$ .

We now turn to the consequences of (1.1). Let  $f : X \rightarrow Y$  be a bundle mapping of a compactum  $X$ . Since  $f$  is an open mapping onto  $Y$  it follows that  $Y, A$  are compacta ( $A = f^{-1}y_0$ ). Let  $A$  be connected, contractible in  $X$  and an  $ANR$ . Since  $A$  is a neighbourhood retract of  $X$  it will certainly be an  $ANR$  if  $X$  is an  $ANR$ . Let  $\dim X < \infty$ . Then  $\dim A, \dim Y \leq \dim X < \infty$  in consequence of the local product representation. Let  $H^*(P, G), H^n(P, G)$  denote the (discrete) Čech cohomology ring and the  $n^{\text{th}}$  Čech cohomology group of a given compactum  $P$ , with coefficients in a given ring  $G$ . We assume that  $Y$  is locally and globally pathwise connected and that  $\pi_1(Y)$ , operating on  $H^*(A, G)$  as in [9], operates simply for every  $G$ . Since  $A$  is connected, and hence pathwise connected, this will certainly be the case if  $X$  is simply connected. For then  $\pi_1(Y) = 1$ . Let  $I_0, R$  and  $S^n$  denote respectively the ring of integers, the ring of rational numbers and an  $n$ -sphere. We write  $H^*(P, I_0) = H^*(P), H^i(P, I_0) = H^i(P)$ . In § 3 we prove :

**Theorem (1.3).** *Let  $H^*(X) \approx H^*(S^n)$  ( $n \geq 1$ ). Then :*

- a) *either  $A$  is an<sup>5)</sup>  $AR$  or  $H^*(A, R) \approx H^*(S^q, R)$ , for some odd value of  $q$ ,*
- b) *if  $A$  is homeomorphic to a topological product,  $A_1 \times A_2$ , then one of  $A_1, A_2$  is an  $AR$ .*

In consequence of the second alternative in (1.3a) we have the exact sequence of Gysin ([3], [9], Ch. III)

$$\dots \xrightarrow{f^*} H^{j-1}(X, R) \rightarrow H^{j-q-1}(Y, R) \xrightarrow{\theta} H^j(Y, R) \xrightarrow{f^*} \dots,$$

in which  $\theta v = v \cup \Omega$  for some  $\Omega \in H^{q+1}(Y, R)$ . Since  $\dim Y < \infty$  there is a  $k \geq 0$  such that  $\Omega^k \neq 0, \Omega^{k+1} = 0$ , where  $\Omega^0 = 1 \in H^0(Y, R), \Omega^r = \Omega \cup \dots \cup \Omega \in H^{r(q+1)}(Y, R)$ . It may be verified that  $k > 0$ , since  $A$  is contractible in  $X$ , that  $n = k(q+1) + q$  and that

$$\left. \begin{array}{ll} H^i(Y, R) \approx R & \text{for } i = 0, q+1, \dots, k(q+1) \\ H^i(Y, R) = 0 & \text{for all other values of } i. \end{array} \right\} \quad (1.4)$$

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<sup>5)</sup> This will be the case, for example, if  $X = Y$  and  $f = 1$ .



Let  $S^n \rightarrow Y$  be a bundle mapping with a connected fibre  $F$ . Then  $F$  is an  $ANR$ , which is contractible in  $S^n$  except in the trivial case  $F = S^n$ . Therefore we have :

**Corollary (1.5).** *If  $F$  is homeomorphic to  $A_1 \times A_2$ , then one of  $A_1, A_2$  is an  $AR$ .*

The results (1.2), (1.5) above extend the two theorems concerning the fibering of Euclidean spaces and spheres by tori which are proved in [4]. Also (1.5) extends a theorem due to A. Borel ([1], [2], p. 165). It will be seen that our (1.3) is an easy corollary of (1.1) together with this theorem of Borel's.

2. Let  $P, Q$  be topological spaces which are either locally finite  $CW$ -complexes or  $ANR$  compacta. Then so are  $P \times Q$  and

$$P \vee Q = (P \times q_0) \cup (p_0 \times Q) \subset P \times Q ,$$

where  $p_0, q_0$  are points in  $P, Q$ , which are 0-cells if  $P, Q$  are  $CW$ -complexes. Let  $P, Q$  be imbedded in  $P \vee Q$  so that  $p = (p, q_0)$ ,  $q = (p_0, q)$  for each point  $p \in P$  and each  $q \in Q$ . Let

$$P \xrightarrow{u} A \xleftarrow{v} Q$$

be given maps such that  $u p_0 = v q_0 = a_0$ , say, and  $u$  is homotopic in  $X$  (and hence homotopic rel.  $p_0$ ) to the constant map  $P \rightarrow a_0$ . Let

$$g_0 : P \times Q \rightarrow X$$

be defined by  $g_0(p, q) = vq$ . Then  $g_0 P = a_0$  and there is a homotopy  $u_t : P \rightarrow X$ , rel.  $p_0$ , such that  $u_0 p = g_0 p = a_0$ ,  $u_1 p = up$  ( $p \in P$ ). This can be extended, first to a homotopy  $u'_t : P \vee Q \rightarrow X$  such that  $u'_t q = vq$  if  $q \in Q$ , and then ([16], p. 228, [15]) to a homotopy  $g_t : P \times Q \rightarrow X$ . Then  $g_1 p = up$ ,  $g_1 q = vq$ . Let

$$h : (P \times Q, P \vee Q) \rightarrow (X, A)$$

be the map determined by  $g_1$ . We describe  $h$  as *inessential* if, and only if, it is related by a homotopy of the form  $(P \times Q, P \vee Q) \rightarrow (X, A)$  to a map with values in  $A$ . We describe  $v$  as *inessential* if, and only if, it is homotopic, and hence homotopic rel.  $q_0$ , to the constant map  $Q \rightarrow a_0$ .

**Lemma (2.1).** *If  $v$  is inessential, so is  $h$ .*

Let  $v_t : Q \rightarrow A$ , rel.  $q_0$ , be a homotopy such that  $v_0 = v$ ,  $v_1 Q = a_0$ . Let  $g' : P \times Q \times I \rightarrow X$  be defined by

$$\begin{aligned}
g'(p, q, t) &= g_{1-3t}(p, q) & \text{if } 0 \leq t \leq 1/3 \\
&= v_{3t-1}(q) & \text{if } 1/3 \leq t \leq 2/3 \\
&= u_{3t-2}(p) & \text{if } 2/3 \leq t \leq 1 .
\end{aligned}$$

Then  $g'(p, q, 0) = g_1(p, q) = h(p, q)$ ,  $g'(Q \times I) \subset A$  and  $g'(P \times Q \times 1) \subset A$ . Also  $g'(p_0 \times q_0 \times I) = a_0$  and since  $g_s(p, q_0) = u_s(p)$  it follows that  $g' \mid (P \vee Q) \times I$  is homotopic, rel.  $(P \times 0) \cup (Q \times I) \cup (P \times 1)$ , to a map in which  $(p, t) \rightarrow up$ . Therefore (2.1) follows from the homotopy extension theorem, applied to the pair  $(P \times Q \times I, K)$ , where

$$K = (P \times Q \times 0) \cup (P \vee Q) \times I \cup (P \times Q \times 1) .$$

Notice that we have used the form of the homotopy extension theorem in which the argument spaces are of a special sort and the image space is arbitrary. The definition of  $h$  and the proof of (2.1) apply unchanged if  $X$  is an  $ANR$ , of the sort appropriate to some general category of spaces to which  $P, Q, P \times Q$  etc. belong (cf. [6]).

3. *Proof of (1.1).* Let  $f' : (X, A) \rightarrow (Y, y_0)$  be the map determined by  $f$ . Then

$$f' \circ h : (P \times Q, P \vee Q) \rightarrow (Y, y_0)$$

is defined in the same way as  $h$ , in § 2, with  $g_t, v$  replaced by  $f \circ g_t$  and the constant map  $Q \rightarrow y_0$ . Therefore it follows from (2.1) that  $f' \circ h$  is homotopic, rel.  $P \vee Q$ , to the constant map  $c$ , where  $c(P \times Q) = y_0$ . Assuming that a homotopy  $f' \circ h \simeq c$  can be lifted it follows that  $h$  is inessential. Therefore  $h \mid P \vee Q$  has an extension  $P \times Q \rightarrow A$  and (1.1) follows on taking  $P = Q = A$  and  $u = v =$  the identical map.

Let  $f$  be a bundle mapping and let  $X$  be a locally compact, separable metric space. Then  $X$  and likewise  $A$  and  $A \times A$  are obviously  $C_\sigma$ -spaces. Therefore we have, in consequence of the concluding remarks in § 2 and § 11.3 in [13]:

**Theorem (3.1).** *If  $f : X \rightarrow Y$  is a bundle mapping, if  $X$  is a locally compact, separable metric  $ANR$  and if a fibre,  $A$ , is contractible in  $X$ , then  $A$  is an  $H$ -space.*

4. *Proof of (1.3).* Let  $g : E \rightarrow B$  be a fibre mapping, with fibre  $F$ , where  $E, B$ , and hence also  $F$ , are compacta. Let  $H^i(P) = 0$  for  $P = B, E, F$  and all sufficiently large values of  $i$ . This will be the case, for example, if  $\dim P < \infty$ . Also let  $H^i(P)$  be finitely generated for all values of  $i$ . It follows from the theory of the spectral sequence associated with the mapping  $g$  that this will be the case if any two of  $H^i(B), H^i(E),$

$H^i(F)$  are finitely generated for every <sup>6)</sup>  $i$ . Therefore it will be the case if  $H^*(E) \approx H^*(S^n)$  and  $F$  is an  $ANR$ . We quote the universal coefficient theorem <sup>7)</sup>

$$H^r(Q, G) \approx H^r(Q) \otimes G + H^{r+1}(Q) * G, \quad (3.1)$$

for the (discrete) Čech cohomology groups of a compactum  $Q$ , with coefficients in  $G$ . It follows from (3.1) that, if

$$H^m(Q) \approx I_0, \quad H^i(Q) = 0 \quad \text{for } i > m, \quad (3.2)$$

then

$$H^m(Q, G) \approx G, \quad H^i(Q, G) = 0 \quad \text{for } i > m. \quad (3.3)$$

Let  $H^j(Q)$  be finitely generated for every  $j \geq m$ . Then (3.2) is true if (3.3) holds for every field,  $G$ , as group of coefficients.

Let  $E$  satisfy (3.2) for some  $m \geq 0$  and let  $K$  be a given field. Then it follows from Theorem (9.1) on p. 189 of [9] that there are integers  $r = r_K$ ,  $s = s_K$  such that  $r + s = m$  and

$$\left. \begin{aligned} H^r(B, K) &\approx K, & H^i(B, K) &= 0 & \text{if } i > r, \\ H^s(F, K) &\approx K, & H^j(F, K) &= 0 & \text{if } j > s, \end{aligned} \right\} \quad (3.4)$$

in which  $\approx$  indicates isomorphism between vector spaces over  $K$ . Let  $k = r_R$ ,  $l = s_R$ . Since  $H^i(B)$ ,  $H^i(F)$  are finitely generated it follows from (3.1) and (3.4), with  $K = R$ , that

$$H^k(B) \approx I_0 + T, \quad H^l(F) \approx I_0 + T',$$

where  $T$ ,  $T'$  are finite groups. Hence, and from (3.1), it follows that  $H^k(B, K)$  and  $H^l(F, K)$  each contains a summand which is isomorphic to  $K$ . Therefore  $k \leq r$ ,  $l \leq s$  and since  $k + l = m = r + s$  we have  $k = r$ ,  $l = s$ . Thus  $r$ ,  $s$  are independent of the choice of  $K$ . Therefore  $B$ ,  $F$  satisfy (3.2) with  $m$  replaced by  $r$  or  $s$  according as  $Q = B$  or  $F$ .

Let  $f: X \rightarrow Y$  and  $A$  be as in (1.3). Then  $H^*(X) = H^*(S^n)$  and it follows from the preceding paragraph that

$$\left. \begin{aligned} \text{a) } H^p(Y) &\approx I_0, & H^i(Y) &= 0 & \text{if } i > p \\ \text{b) } H^q(A) &\approx I_0, & H^j(A) &= 0 & \text{if } j > q \end{aligned} \right\} \quad (3.5)$$

<sup>6)</sup> The argument is essentially the same as the one on p. 465 of [12]. See also § 9 of [9]

<sup>7)</sup> See Theorem 44.2 on p. 823 of [5], in which the term  $H^{r+1}(Q) * G$  is expressed differently. The only property of this "product" which we need is that  $H * G = 0$  if either  $H = 0$  or if  $G$  has no (non-zero) element of finite order. We use  $+$  to indicate direct summation.

for some pair of integers  $p, q$  such that  $p + q = n$ . Moreover  $A$ , being contractible in  $X$ , is an  $H$ -space, according to (1.1).

First assume that  $q = 0$ . Then it follows from (5.1) on p. 346 of [10] that  $H_i(A) = 0$  for every  $i > 0$ , where  $H_i(A)$  is the  $i^{\text{th}}$  discrete, integral Čech homology group of  $A$ . Since  $A$  is an  $ANR$  it follows that  $H'_i(A) = 0$  if  $i > 0$  where  $H'_i(A)$  is the  $i^{\text{th}}$  singular homology group of  $A$  ([11], p. 107). Hence it follows, as in the proof of (1.2), that  $A$  is an  $AR$ . If  $A$  is homeomorphic to  $A_1 \times A_2$ , then  $A_1$  is homeomorphic to a retract of  $A$ . Therefore it follows that  $A_1$  is an  $AR$ . This proves (1.3) if  $q = 0$  and we proceed on the assumption that  $q > 0$ .

Since  $A$  is an  $H$ -space and  $H^j(A) = 0$  if  $j > q$  we have ([8], No. 24)

$$H^*(A, R) \approx H^*(S^{i_1} \times \dots \times S^{i_p}, R)$$

for certain odd values of  $i_1, \dots, i_p$ . Since  $H^*(X) \approx H^*(S^n)$  it follows from [1] and [2], p. 165, that  $p = 1$ . Thus

$$H^*(A, R) \approx H^*(S^q, R), \quad (3.6)$$

where  $q$  is odd. This proves (1.3a).

Let  $A$  be homeomorphic to  $A_1 \times A_2$ . On taking  $g : E \rightarrow B$  to be the projection  $A_1 \times A_2 \rightarrow A_2$ , with  $F$  homeomorphic to  $A_1$ , it follows from (3.5b) and (3.4) that

$$H^{q_j}(A_j) \approx I_0, \quad H^i(A_j) = 0 \quad \text{if} \quad i > q_j,$$

for some pair of integers  $q_1, q_2$  such that  $q_1 + q_2 = q$ . The group  $H^{q_j}(A_1 \times A_2, R)$  contains a summand which is isomorphic to  $H^{q_j}(A_j, R)$ . Hence it follows from (3.6) that  $q_j = 0$  or  $q$ . Since  $q_1 + q_2 = q$  it follows that either  $q_1 = 0$  or  $q_2 = 0$ , say  $q_1 = 0$ . Since  $A_1$  is homeomorphic to a retract of  $A$  it is an  $ANR$ . Since  $\pi_1(A) \approx \pi_1(A_1) \times \pi_1(A_2)$  and  $\pi_1(A)$  is Abelian, because  $A$  is contractible in  $X$ , it follows that  $\pi_1(A_1)$  is Abelian. Therefore it follows from an argument similar to the one used above to dispose of the case  $q = 0$  that  $A_1$  is an  $AR$ . This completes the proof.

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