Expansion of the Hypergeometric Function in Series of Confluent Ones and Application to the Jacobi Polynomials.

Autor(en): **Tricomi, Francesco G.**

Objekttyp: **Article**

Zeitschrift: **Commentarii Mathematici Helvetici**

Band (Jahr): **25 (1951)**

PDF erstellt am: **10.05.2024**

Persistenter Link: https://doi.org/10.5169/seals-20703

Expansion of the Hypergeometric Function in Series of Confluent Ones and Application to the Jacobi Polynomials[1)]

By Francesco G. Tricomi

1. The corner-stone of this paper is the use of an expansion of the confluent hypergeometric function:

$$\Phi(a, c; x) \equiv {}_1F_1(a; c; x) = \sum_{m=0}^{\infty} \frac{(a)_m}{(c)_m} \frac{x^m}{m!}$$

in series of Bessel functions, which I first gave in 1941 in the case of the Laguerre polynomials[2)] and which I later on generalized and improved[3)].

If we introduce the useful abbreviation

$$E_\nu(x) \equiv x^{-\nu/2} J_\nu(2\sqrt{x}) = \sum_{m=0}^{\infty} \frac{(-x)^m}{m!\,\Gamma(\nu+m+1)} \tag{1}$$

and designate by $A_m(k, l)$ the coefficients given by the generating function[4)]

$$\sum_{m=0}^{\infty} A_m(k, l)\, z^m = e^{2kz}(1-z)^{k-l}(1+z)^{-k-l}\,, \qquad (|z|<1) \tag{2}$$

the said expansion can be put into the simple form

$$\frac{e^{-x/2}}{\Gamma(c)} \Phi(a, c; x) = \sum_{m=0}^{\infty} A_m\left(k, \frac{c}{2}\right)\left(\frac{x}{2}\right)^m E_{c+m-1}(kx)\,, \qquad k = \frac{c}{2} - a \tag{3}$$

In the papers quoted I have above all emphasized the asymptotic character of the previous expansion as $k \to \infty$ but I have also shown that the series converges even in the usual sense *at least* as long as

---

1) Research sponsored by the Office of Naval Research.

2) Sviluppo dei polinomi di Laguerre e di Hermite in serie di funzioni di Bessel, Giorn. Ist. Italiano Attuari *12* (1941), 14—33.

3) Sulle funzioni ipergeometriche confluenti, Ann. Mat. Pura Appl. (4) *26* (1947—1948), 141—175; Sul comportamento asintotico dei polinomi di Laguerre, Ibidem (4) *28* (1949), 263—289.

4) These coefficients are thoroughly studied in my paper: A class of non-orthogonal polynomials related to these of Laguerre, in progress of printing in the new Journal d'Analyse Mathématique (Jerusalem).

$0 \leq x < 4k$. Now I have seen that, better still, *the series* (3) *is always convergent in the entire x-plane.*

To prove this it is sufficient to proceed in a very similar manner to the method used in the case of the Neumann series[5]).

In fact, since

$$E_\nu(x) = \frac{1}{\Gamma(\nu+1)}[1 + O(|\nu|^{-1})], \qquad \frac{m!}{\Gamma(m+\alpha)} = m^{1-\alpha}[1 + O(m^{-1})],$$

given a series of the form

$$\sum_{m=0}^{\infty} a_m z^m E_{\alpha+m}(z) \tag{4}$$

we have

$$|a_m E_{\alpha+m}(z)|^{1/m} = \left|\frac{a_m}{m!}\right|^{1/m} e^{(1-\alpha)\log m/m}[1 + O(m^{-1})]^{1/m}$$

and consequently

$$\overline{\lim_{m\to\infty}}\, |a_m E_{\alpha+m}(z)|^{1/m} = \overline{\lim_{m\to\infty}} \left|\frac{a_m}{m!}\right|^{1/m}$$

and this shows that the series (4) has the same circle of convergence as the *associated power series*

$$\sum_{m=0}^{\infty} \frac{a_m}{m!} z^m . \tag{5}$$

In particular the series (3) is *always convergent* because the power series

$$\sum_{m=0}^{\infty} A_m(k, l) \frac{z^m}{m!}$$

has an infinite radius of convergence since the series without $m!$, i. e. the series (2), has the radius of convergence unity.

We notice further that the coefficients A_m satisfy the recurrence relation

$$(m+1)A_{m+1} = (m + 2l - 1)A_{m-1} - 2kA_{m-2}, \quad (m = 2, 3, \ldots) \tag{6}$$

and that the first of them are:

$$A_0 = 1, \; A_1 = 0, \; A_2 = l, \; A_3 = -\frac{2}{3}k, \; A_4 = \frac{l(l+1)}{2},$$
$$A_5 = -2k\left(\frac{l}{3} + \frac{1}{5}\right), \ldots \tag{7}$$

2. If we consider that the ordinary hypergeometric function $F(a, b; c; x)$ can be expressed as the Laplace transform of the function Φ, that is

---

[5]) *G. N. Watson*, Bessel functions, § 16.2.

$$F\left(a,b;c;\frac{1}{s}\right)=\frac{s^b}{\Gamma(b)}\,\mathfrak{L}_s[t^{b-1}\Phi(a,c;t)]\ ,\qquad (\Re b>0\,,\ \Re s>1) \tag{8}$$

it immediately occurs to us to seek for F an expansion analogous to (3).

With a term-by-term integration we have *at least formally*

$$F\left(a,b;c;\frac{1}{s}\right)=\frac{\Gamma(c)}{\Gamma(b)}\,s^b\sum_{m=0}^{\infty}A_m\left(k,\frac{c}{2}\right)2^{-m}\,\mathfrak{L}_s\,[e^{t/2}\,t^{b+m-1}E_{c+m-1}(k\,t)]=$$
$$=\frac{\Gamma(c)}{\Gamma(b)}\,s^b\sum_{m=0}^{\infty}A_m\left(k,\frac{c}{2}\right)2^{-m}\,\mathfrak{L}_{s-1/2}\,[t^{b+m-1}E_{c+m-1}(k\,t)]\ .$$

But

$$\mathfrak{L}_{s-1/2}\,[t^{b+m-1}E_{c+m-1}(k\,t)]=\frac{\Gamma(b+m)}{\Gamma(c+m)}\left(s-\frac{1}{2}\right)^{-(b+m)}\Phi\left(b+m,c+m;-\frac{k}{s-\frac{1}{2}}\right)$$

hence, putting $s=k/z+1/2$ and with the help of Kummer's formula and further simple transformations we obtain (for $k=c/2-a$)

$$\boxed{F\left(a,b;c;\frac{2z}{2k+z}\right)=\left(1+\frac{z}{2k}\right)^b e^{-z}\sum_{m=0}^{\infty}\frac{(b)_m}{(c)_m}A_m\left(k,\frac{c}{2}\right)\left(\frac{z}{2k}\right)^m\Phi(c-b,c+m;z)\,.} \tag{9}$$

This result is not merely a formal one because the series on the rigth side converges like a power series as long as

$$|\,z\,|<2\,|\,k\,|\ . \tag{9'}$$

In fact, for reasons similar to those above, a series of the form

$$\sum_{m=0}^{\infty}a_m\,z^m\,\Phi(a\,,c+m\,;z)$$

has the same circle of convergence as the associated power series $\Sigma\,a_m z^m$ [6]) and in our case the power series

$$\sum_{m=0}^{\infty}\frac{(b)_m}{(c)_m}\,\frac{A_m(k\,,c/2)}{(2\,k)^m}\,z^m$$

has obviously the radius of convergence $2\,|\,k\,|$.

However this is not yet sufficient to justify the sign $=$ between both sides of (9) but this difficulty can be eliminated by observing: i) that we have surely $|\,A_m\,|\leq 1$ provided that m is large enough; ii) that — at least if k and c are real and positive — by grace of a well-known inequality [7]) we have

---

[6]) *A. Erdélyi*, Funktionalrelationen mit konfluenten hypergeometrischen Funktionen, II., Math. Zeitschr. *42* (1937), 641—670 (§ 10, p. 665).

[7]) *Watson*, op. cit. p. 49, form. (1).

$$| E_{c+m-1}(k t) | \leqslant \frac{1}{\Gamma(c+m)} .$$

Consequently instead of the still doubtful (9) we can write

$$F\left(a, b; c; \frac{2z}{2k+z}\right) = \left(1+\frac{z}{2k}\right)^{b} e^{-z} \sum_{m=0}^{n-1} \frac{(b)_m}{(c)_m} A_m\left(k, \frac{c}{2}\right)\left(\frac{z}{2k}\right)^{m} \Phi(c-b, c+m; z) +$$
$$+ \frac{\Gamma(c)}{\Gamma(b)}\left(\frac{k}{z}+\frac{1}{2}\right)^{b} \mathfrak{L}_{k/z+1/2}\left[R_n(t)\right]$$

where the function

$$R_n(t) = e^{t/2} t^{b-1} \sum_{m=n}^{\infty} A_m\left(k, \frac{c}{2}\right)\left(\frac{t}{2}\right)^{m} E_{c+m-1}(k t)$$

if n is large enough and b is also real, is such that

$$| R_n(t) | \leqslant e^{t/2} t^{b-1} \sum_{m=n}^{\infty} \frac{(t/2)^m}{\Gamma(c+m)} < e^{t/2} t^{b-1} \frac{(t/2)^n}{\Gamma(c+n)} \sum_{h=0}^{\infty} \frac{(t/2)^h}{h!} = e^{t} \frac{t^{b+n-1}}{2^n \Gamma(c+n)} .$$

But

$$\mathfrak{L}_{k/z+1/2}\left[e^{t} \frac{t^{b+n-1}}{2^n \Gamma(c+n)}\right] = \frac{\Gamma(b+n)}{\Gamma(c+n)} \left(\frac{2z}{2k-z}\right)^{b-1}\left(\frac{z}{2k-z}\right)^{n}$$

as $n \to \infty$ approaches *zero* as long as $|z| < |2k - z|$, i. e. as long as $\Re z < k$; hence the expansion (9) is valid at least in the *left part* of the circle $|z| < 2k$.

Moreover, considering that inside the circle $|z| < 2|k|$ both sides of (9) are analytic functions of z and of the parameters, this last condition as well as the condition of the reality of c, k, b and the conditions $\Re b > 0$, $\Re(2k/z) > 1$ arising from (8) can be disregarded because of the principle of analytic continuation. The only remaining restriction is (9′) and that c does not coincide with $0, -1, -2, \ldots$, a restriction which can also be disregarded if both sides of (9) are divided by $\Gamma(c)$.

3. The importance of the expansion (9) arises principally from the fact that the order of magnitude of its successive terms decreases rapidly if $|k|$ is large compared with $|z|$, a circumstance which is realized in many important applications.

Moreover we can utilize (3) again to expand the confluent functions on the right side of (9) by means of Bessel functions and we find thus the further expansion

$$F\left(a, b; c; \frac{2z}{2k+z}\right) = \left(1+\frac{z}{2k}\right)^{b} e^{-z/2} \sum_{p=0}^{\infty} C_p(z) \left(\frac{z}{2}\right)^{p}, \qquad (|z| < 2|k|) \qquad (10)$$

where

$$\left.\begin{aligned} C_p(z) = \Gamma(c) \sum_{m=0}^{p} \frac{(b)_m}{k^m} A_m\left(k, \frac{c}{2}\right) A_{p-m}\left(k' + \frac{m}{2}, \frac{c+m}{2}\right) E_{c+p-1}\left[\left(k' + \frac{m}{2}\right) z\right] \\ k = \frac{c}{2} - a, \qquad k' = b - \frac{c}{2}. \end{aligned}\right\} \tag{11}$$

In particular we have

$$C_0(z) = \Gamma(c) E_{c-1}(k' z) = 1 - \frac{k'}{c} z + \cdots, \qquad C_1(z) \equiv 0$$

$$C_2(z) = \frac{\Gamma(c+1)}{2} \left\{ E_{c+1}(k' z) + \frac{b(b+1)}{k^2} E_{c+1}[(k'+1) z] \right\}.$$

4. In a previous paper I gave a general method to deduce from an asymptotic representation of a certain function a corresponding representation of its zeros [8]). For instance, from the expansion (3) we can infer to each zero $j_{c-1,r}$ $(r = 1, 2, \ldots)$ of the Bessel function $J_{c-1}(x)$ there corresponds a zero ξ_r of the confluent hypergeometric function $\Phi(a, c; \xi)$ such that we have [9])

$$\xi_r = \frac{j^2_{c-1,r}}{4k} \left[1 + \frac{2c(c-2) + j^2_{c-1,r}}{48 k^2} \right] + O(k^{-5}). \tag{12}$$

In a similar manner we can obtain useful asymptotic expressions as $k \to \infty$ for the zeros in the neighborhood of the origin of the hypergeometric function

$$F\left(a, b; c; \frac{2z}{2k+z}\right) \tag{13}$$

starting from the expansion (9) which leads us to the equation

$$\Phi_{00}(z) + \frac{(b)_2}{(c)_2} A_2\left(k, \frac{c}{2}\right)\left(\frac{z}{2k}\right)^2 \Phi_{02}(z) + \frac{(b)_3}{(c)_3} A_3\left(k, \frac{c}{2}\right)\left(\frac{z}{2k}\right)^3 \Phi_{03}(z) + \cdots = 0 \tag{14}$$

where use is made of the abbreviation

$$\Phi_{nm}(z) = \Phi(c - b + n, c + m; z). \qquad (n, m = 0, 1, 2, \ldots).$$

Now we consider: i) that from the formulae (7) it follows that A_2 and A_4 are $O(1)$ as $k \to \infty$, A_3 and A_5 are $O(k)$, and so forth; ii) that

---

[8]) Sugli zeri delle funzioni di cui si conosce una rappresentazione asintotica, Ann. Mat. Pura Appl. (4) *26* (1947—1948), 283—300.

[9]) This formula is given (with the remainder $O(k^{-4})$ instead of $O(k^{-5})$) as new in my previous paper but it is already contained in *H. Schmidt,* Über Existenz und Darstellung impliziter Funktionen bei singulären Anfangswerten, Math. Zeitschr. *43* (1938), 533—552.

from well-known relations about *contiguous* confluent hypergeometric functions it follows

$$z\,\Phi_{02}(z) = \frac{(c+1)(c+z)}{b(b+1)}\,[c\,\Phi_{00}(z) - (c-b)\,\Phi_{11}(z)]\ ,$$

$$z^2\,\Phi_{03}(z) = \frac{(c+1)_2}{(b)_3}\,[c(c+1) + (2c-b)z + z^2]\,[c\,\Phi_{00}(z) - (c-b)\,\Phi_{11}(z)]\ .$$

$$- \; - \; - \; - \; -$$

We see thus that, under the hypothesis

$$b = O(k)\ , \qquad c = O(1)\ , \qquad z = O(k^{-1})\ , \qquad (k \to \infty) \qquad (15)$$

the second and the third terms of the equation (13) multiplied by $c/(c-b)$ are $O(k^{-3})$, while the fourth and the following ones are at least $O(k^{-5})$. This shows us that equation (14) can be put into the simplified form

$$g_0(z) + g_1(z)\frac{1}{(2k)^2} + O(k^{-5}) = 0 \qquad (16)$$

where

$$g_0(z) = \frac{c}{c-b}\,\Phi_{00}(z)\ , \qquad g_0'(z) = \Phi_{11}(z)\ ,$$

$$g_1(z) = \frac{cz}{c-b}\left[\frac{c}{2}\,\frac{(b)_2}{(c)_2}\,z\,\Phi_{02}(z) - \frac{1}{3}\,\frac{(b)_3}{(c)_3}\,z^2\,\Phi_{03}(z)\right] =$$

$$= -\frac{z}{6}\,[c(c-2) + 2k'z - 2z^2]\left[\Phi_{11}(z) - \frac{c}{c-b}\,\Phi_{00}(z)\right]\ .$$

Now we apply the method of my paper and we obtain

$$z_r^* = z_r - \frac{g_1(z_r)}{g_0'(z_r)}\,\frac{1}{(2k)^2} + O(k^{-5})$$

where z_r and z_r^* are two corresponding zeros of

$$\Phi_{00}(z) \equiv \Phi(c-b, c\,; z) \qquad (17)$$

and the function (13) respectively. Better still, since

$$\frac{g_1(z_r)}{g_0'(z_r)} = -\frac{z_r}{6}\,[c(c-2) + 2k'z_r - 2z_r^2] = -\frac{z_r}{6}\,[c(c-2) + 2k'z_r] + O(k^{-3})$$

we obtain the formula

$$z_r^* = z_r\left[1 + \frac{c(c-2) + 2k'z_r}{24k^2}\right] + O(k^{-5})\ . \qquad (18)$$

Finally we identify z_r with the ξ_r of formula (12) noting that k must be changed into the k' given by the last formula (11), which under our hypothesis is of the same order of magnitude as k. Moreover this ξ_r satisfies the last condition (15). We obtain thus the very simple formula

$$\left.\begin{aligned} z_r^* &= \frac{j_{c-1,r}^2}{4k}\left[1+\frac{2c(c-2)+j_{c-1,r}^2}{48k^2}\right]\left[1+\frac{2c(c-2)+j_{c-1,r}^2}{48k'^2}\right]+O(k^{-5}) \\ k &= \frac{c}{2}-a\,, \quad k'=b-\frac{c}{2}\,; \quad b=O(k)\,, \quad c=O(1)\,, \quad k\to\infty\,. \end{aligned}\right\} \tag{19}$$

From this z_r^*, we deduce a corresponding zero of $F(a, b; c; \zeta)$ putting

$$\zeta_r^* = \frac{2z_r^*}{2k+z_r^*}\,, \tag{20}$$

which implies $\zeta_r^* = O(k^{-2})$.

5. Among other things the previous results can be applied to the asymptotic study of the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ and its zeros in the neighborhood of the end-points of the basic interval $(-1, 1)$, a problem about which very little seems to be known as yet[10]). In fact, by putting

$$x = 1-\frac{4z}{2k+z} \tag{21}$$

we have

$$P_n^{(\alpha,\beta)}(x) = \frac{(\alpha+1)_n}{n!}F\left(-n, n+\alpha+\beta+1, \alpha+1; \frac{2z}{2k+z}\right) \tag{22}$$

the conditions (15) are satisfied as $n\to\infty$ and from (9), (18) and (19) we get respectively

$$P_n^{(\alpha,\beta)}(x) =$$

$$\frac{1}{n!}\left(\frac{1+x}{2}\right)^{-n'}e^{-z}\sum_{m=0}^{\infty}\frac{\Gamma(n'+m)\Gamma(\alpha+n+1)}{\Gamma(n')\Gamma(\alpha+m+1)}A_m\left(k,\frac{\alpha+1}{2}\right)\left(\frac{z}{2k}\right)^m\Phi(-n-\beta, \alpha+m+1; z) \tag{23}$$

with

$$k = n+\frac{\alpha+1}{2}\,, \qquad n' = n+\alpha+\beta+1\,; \qquad |z|<2|k|\,;$$

---

10) In *Szegö*'s Orthogonal Polynomials we find only (p. 186) the formula

$$\lim_{n\to\infty} n^{-\alpha} P_n^{(\alpha,\beta)}[1-z/(2n^2)] = (z/2)^{-\alpha} J_\alpha(z)$$

and (p. 189) Hilb's formula for the Laguerre Polynomials

$$P^n(\cos\theta) = \sqrt{\theta/\sin\theta}\, J_0[(n+\tfrac{1}{2})\theta] + O(n^{-3/2})$$

together with its generalisation for $P_n^{(\alpha,\beta)}$ (p. 191). Moreover, *MacDonald*, Proc. London Math. Soc. (2) *13* (1914), 220—221 has given another, more complicated and not completely convincing, extension of the same Hilb's formula. For a different kind of representation of $P_n(\cos\theta)$ by means of Bessel functions, see the paper of *G. Szegö* in the same Proceedings (2) *36* (1937), 427—450.

$$z_r^* = z_r\left(1 + \frac{\alpha^2 - 1 + 2k' z_r}{24k^2}\right) + O(n^{-5}) \qquad (24)$$

with

$$k' = n + \frac{\alpha+1}{2} + \beta = k + \beta\,, \qquad \Phi(-n-\beta, \alpha+1; z_r) = 0$$

and

$$z_r^* = \frac{j_{\alpha,r}^2}{4k}\left[1 + \frac{2(\alpha^2-1) + j_{\alpha,r}^2}{48k^2}\right]\left[1 + \frac{2(\alpha^2-1)+j_{\alpha,r}^2}{48k'^2}\right] + O(n^{-5})\,. \qquad (25)$$

In particular if $\beta = 0$ the confluent function Φ reduces to a Laguerre polynomial and with the usual notations for these we have

$$P_n^{(\alpha,0)}(x) = \left(\frac{1+x}{2}\right)^{-n'} e^{-z} \sum_{m=0}^{\infty} A_m\left(k, \frac{\alpha+1}{2}\right)\left(\frac{z}{2k}\right)^m L_n^{(\alpha+m)}(z)\,,$$

$$(|z| < 2|k|)\,. \qquad (26)$$

and

$$z_r^* = \lambda_{n,r}^{(\alpha)}\left[1 + \frac{\alpha^2-1+\lambda_{n,r}^{(\alpha)}}{24k^2}\right] + O(n^{-5}) = \frac{j_{\alpha,r}^2}{4k}\left[1+\frac{2(\alpha^2-1)+j_{\alpha,r}^2}{48k^2}\right]^2 + O(n^{-5})\,. \qquad (27)$$

In the special case $\alpha = 0$ we get the corresponding formulae for the Legendre polynomials to which we may add the expansion (10), which in this case yields the formula

$$P_n(x) = \\ = \left(\frac{4}{x+3}\right)^{n+1} e^{-\xi/(2n+1)}\left\{J_0(2\sqrt{\xi}) + \frac{\xi}{2(2n+1)^2}\left[J_2(2\sqrt{\xi}) + \frac{(2n+2)(2n+4)}{(2n+1)(2n+3)} J_2\left(2\sqrt{\frac{2n+3}{2n+1}\xi}\right)\right] + O(n^{-4})\right\}$$

with (28)

$$x = 1 - \frac{8\xi}{(2n+1)^2 + 2\xi}\,, \qquad \xi = \frac{(2n+1)^2}{2}\,\frac{1-x}{3+x}\,.$$

With a remainder $O(n^{-3})$ instead of $O(n^{-4})$ we can write more simply

$$P_n(x) = \left(\frac{4}{x+3}\right)^{n+1} e^{-\xi/(2n+1)}\left[J_0(2\sqrt{\xi}) + \frac{\xi}{8n^2} J_2(2\sqrt{\xi}) + O(n^{-3})\right]\,. \qquad (29)$$

6. To show the utility of the previous formulae we will use (27) for the numerical evaluation of the two least zeros x_1 and x_2 of the Legendre polynomial $P_{10}(x)$, whose exact values (with seven figures) are[11])

$$x_1 = 0.973\,9065\,, \qquad x_2 = 0.865\,0634\,.$$

[11]) *A. N. Lowan, N. Davids, A. Levenson,* Table of the zeros of the Legendre polynomials ..., Bull. Amer. Math. Soc. *48* (1942), 739—743 and *49* (1943), 939 (errata).

With the first formula (27), using the values of the first two zeros of the Laguerre polynomial $L_{10}(x)$ given by *H. E. Salzer* and *R. Zucker*[12]), we find

$$x_1' = 0.973\,9092 \;, \qquad x_2' = 0.865\,0183 \;.$$

while the second formula (27) gives us

$$x_1'' = 0.973\,9092 \;, \qquad x_2'' = 0.865\,0308 \;.$$

The agreement is very good, especially for the last zero x_1.

On the contrary if use is made of the very simple asymptotic formula

$$x_r^* = \left(1 - \frac{1}{8n^2} + \frac{1}{8n^3}\right) \cos\left(\frac{4r-1}{4n+2}\pi\right) + O(n^{-4}) \;, \quad \left[r - \frac{n}{2} = O(1)\right] \tag{30}$$

given in one of my last papers[13]) we have

$$x_1^* = 0.973\,8309 \;, \qquad x_2^* = 0.865\,0512$$

which is better for x_2 but less good for x_1, in accordance with the fact that the formula (30) is especially suitable for the evaluation of the *central* zeros of $P_n(x)$.

(Eingegangen den 17. Juli 1950.)

---

[12]) Table of the zeros and weight factors of the first fifteen Laguerre polynomials, Ibidem *55* (1949), 1004—1012.

[13]) Sugli zeri dei polinomi sferici ed ultrasferici, Ann. Mat. Pura Appl. (4) *31* (1950), 93—97.