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Dynamic Finite Element Analysis of Reinforced Concrete Structures

Analyse dynamique de structures en béton armé à l'aide d'éléments finis

Dynamische Finite Elemente Analysis von Stahlbetontragwerken

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SUMMARY
This report summarizes the current state of the art in finite element and other numerical techniques applied to concrete structures subjected to dynamic loads. The various sources of dynamic loads are identified, followed by a brief statement on strain-rate effects on material properties. The behavior of reinforced concrete members under cyclic loading is discussed, as well as some of the mathematical models devised to sumulate this behavior. The last section considers some of the applicable dynamic analysis methods.

RESUME
Cet article résume l'état actuel des connaissances sur les éléments finis et autres méthodes numériques appliquées aux structures en béton armé soumises à des charges dynamiques. Les différentes origines des charges dynamiques sont établies; un bref exposé des effets de la vitesse de déformation sur les propriétés des matériaux est présenté. Le comportement des poutres en béton armé soumises à un cycle de charges et quelques modèles mathématiques destinés à la simulation du comportement sont discutés. La dernière partie traite quelques méthodes d'analyse dynamique.

ZUSAMMENFASSUNG
Dieser Bericht fasst den heutigen Stand der Finite-Element- und anderer numerischer Methoden zusammen, welche auf dynamisch beanspruchte Stahlbetontragwerke angewendet werden. Die verschiedenen Ursprünge der dynamischen Lasten werden aufgeführt, gefolgt von einer Bemerkung über den Einfluss der Dehnungsgeschwindigkeit auf die Materialeigenschaften. Das Verhalten von Stahlbetonstäben unter zyklischer Belastung wird diskutiert, sowie einige der mathematischen Modelle, welche dieses Verhalten simulieren sollen. Der letzte Abschnitt behandelt einige Anwendungsmethoden für dynamische Analysen.

1. INTRODUCTION

It is the responsibility of the structural engineer to ensure that all structures are accurately analyzed and designed to safely withstand the anticipated loads. Most structures are designed on the basis of relatively simple methods which are the result of mostly empirical research compiled over many decades and which have been confirmed by innumerable experimental tests. This traditional approach to design has generally been adequate because

1. the structures designed by these methods have by and large behaved satisfactorily, and
2. the extraordinary complex behavior of concrete (an inelastic, viscous, inhomogeneous material) has stymied any attempts at rational formulations to serve as foundation for alternate design methods.

On both counts, considerable changes have taken place in recent years.

The adequacy of traditional empirical design methods has been questioned in a number of applications. Foremost is the construction of nuclear power plants. The overall design requirement that the uncontrolled release of radioactivity be prevented at any rate places demands on the structural analyst and designer that cannot be met by the simplified traditional methods.

A second application for which the standard design methods are clearly inadequate is the design of concrete structures to safely withstand earthquake ground motions. The destructiveness of strong earthquakes is the driving force behind the search for advanced methods of analysis and design that are based on thorough understanding of reinforced concrete behavior.

A third class of structures for which the old design methods are not satisfactory can be broadly labeled as defense installations, such as hardened shelters and missile silos. Their unusual loadings and performance requirements set this class of structures apart from all others.

As the above examples illustrate, there is no reason to feel complacent about currently available design methodology--we have to search for new improved methods. Advanced mechanics in reinforced concrete provides the foundation for these methods. Three separate and yet interrelated tasks can be distinguished:

1. to identify the material properties of concrete and steel;
2. to model the behavior of reinforced concrete members;
3. to analyze the structural response of concrete structures.

This report will summarize the current state of the art in finite element and other numerical techniques applied to concrete structures. While the companion introductory report by Argyris et al emphasizes primarily quasi-static loadings, this report will concentrate on the special problems associated with dynamic loads.

After a brief discussion of the different kinds of dynamic loads, the report will be organized along the three tasks mentioned above. First, the dynamic effects of loading rates on the material properties are discussed briefly. Thereafter, the behavior of reinforced concrete members under cyclic loading will be summarized, as well as some of the mathematical models that have been devised in the past for the purpose of simulating this behavior. The last section will consider some analysis methods that are appropriate for the response analysis of concrete structures to dynamic loads.

2. DYNAMIC LOADS

There are few loads that are truly static in nature, although many types of loadings are applied slowly enough not to excite any significant dynamic response. Still, most loadings that are of concern to the designing engineer because of their effect on the safety or serviceability of structures are of dynamic nature. Below follows a summary description of the important dynamic load categories and their main effects on structures.

2.1 Earthquakes

Most earthquakes occur when the stresses in the border regions of tectonic plates exceed the strength of the rock materials and the vast amounts of stored strain energy are suddenly released, radiating by wave propagation through the earth [1]. Often, the earthquake causes ground rupture in the direct vicinity of the causative fault, above the epicentrum. Although such ground rupture zone may extend for hundreds of miles along the fault in a large earthquake, the directly affected area is normally small. Yet, it is next to impossible under economic constraints, to design structures to safely withstand ground rupture.

The much more important effect of an earthquake is the ground shaking that may set hundreds or thousands of square miles in motion. Of this effect, the structural engineer is concerned more with the intensity of ground shaking than with the Richter magnitude, because it is the intensity which is the primary measure of the effect that the earthquake has on the response of structures.

The ground motion effect is equivalent to inertia loads applied to the various mass points of the structure with its base fixed. These inertia loads are equal to the masses multiplied with the ground acceleration time history.

The importance of earthquake loadings for reinforced concrete structures lies in the fact that reinforced concrete members suffer varying degress of stiffness degradation if subjected to cyclic loading into the inelastic range. It is this decrease in stiffness, coupled with severe cracking of the concrete and yielding of flexural and shear reinforcement that poses the main difficulties when analyzing concrete structures subjected to strong earthquake ground motions. There is no evidence, however, that the load-carrying mechanisms are affected appreciably by the dynamic nature of the loads. In other words, provided the effective inertia loads are correctly simulated, equivalent slow-motion load tests should cause similar structure responses. Thus, the reinforced concrete models developed for statically applied loads can be used also for dynamic loads due to earthquakes.

2.2 Wind

Atmospheric disturbances are ubiquitous on earth and therefore all above-ground structures have to be designed to resist the resulting wind loads [2]. These may be associated with rather steady air movements which can be simulated with statically applied equivalent pressures, or they may be caused by buffeting of truly dynamic nature. The intensities vary widely, from those due to moderate gusts, up to those accompanying tornadoes and hurricanes. While earthquake ground accelerations are characterized by vibratory motion about a zero base line, wind can be considered to consist of pressure fluctuations about a static lateral pressure associated with a mean velocity. This decompositon has the advantage that the mean velocity component can be treated as a static load, while the remaining oscillatory component can be analyzed by similar analysis techniques as earthquake effects. When superimposing the two components, attention has to be paid to the nonlinearity of concrete response.

As far as the dynamic effect of wind loading on reinforced concrete is concerned, a statement similar to that on earthquakes can be made: strain rates induced by wind loadings are not high enough to have a marked effect on the response behavior so that any material models can be used for dynamic analysis as if wind loads are applied very slowly, provided, of course, that the inertia effects are duly accounted for.

2.3 Ocean Waves

Ever since the limited world reserve of fossil fuels started to affect entire national economies, the exploration of the seas for oil became an economical necessity and the construction of concrete drilling platforms a reality. The loadings imposed on these structures by the sea are truly remarkable, since some of the richest off-shore oil fields are situated in the most punishing environments on earth. The loadings consist of extraordinary surface wave forces as well as below-surface water currents. The resulting pressures and drag forces that act on the structure have to be included in any analysis and their dynamic effects must be considered for design.

Earthquakes, wind and ocean waves all have in common that the load intensities and frequency contents are highly random quantities. Single deterministic analyses are therefore of only limited value, and the design for unique deterministic loads raises questions about the reliability of the structure in withstanding future events.

2.4 Missile Impact

The problem of missile impact arises in a number of situations in which reinforced concrete walls or roofs serve as protection against missiles such as tornado-borne debris, aircraft impact, industrial missiles such as fragments of a fractured turbine, high-energy pipe ruptures, or against penetrators used in military attack. Such missile barriers are common design features in safety-class structures of nuclear power plant facilities, as well as in hardened military shelters.

For many of these design problems, the empirical methods proposed in the literature are not adequate. More refined numerical simulation techniques are needed to predict the target response. It is the objective of such numerical calculations to determine the depth of penetration or the residual velocity of the missile if penetration occurred, whether concrete fragments come off the front or real surface, or whether a plug forms in the concrete.

At the high missile velocities considered here, concrete properties, especially tensile strength, are strongly strain-rate dependent. Numerical analysis programs designed to simulate missile impact must therefore be general enough to incorporate a most general material model.

2.5 Blast

Blast loads are encountered primarily in the field of weapons effects on protective structures. The predominant effects are caused by airblast or by shock waves propagating through the ground or by a combination of both. Special cases include explosions on the surface of a concrete structure in which the load intensity depends on the mass of the explosive casing and detonation products.

The main effects of airblast are due to dynamic overpressure and reflected pressure. An example of overpressure loading is a surface-flush structure whose roof is exposed to a shock wave propagating in air. An example of reflected pressure loading is an above-ground structure having a dimension perpendicular to the direction of shock propagation; these are virtually all above-ground structures. The parameters governing the pressure and impulse delivered to the target include the explosive yield, height-of-burst, range of the target from the lowest point, and environmental factors such as atmospheric dust. Although conclusive evidence is lacking, it is generally considered that brittle modes of failure such as shear failures can be explained in terms of peak pressure. In contrast, ductile failure modes such as flexural failure are thought to depend heavily on impulse.

All structures sited within a critical range of an explosion are subjected to ground shock. Attenuation of shock intensity due to geometric dispersion and hysteresis energy absorption is very significant and extremely complicated due to the complexity of in-situ properties of rock and soil. Direct or crater-induced shock begins at or beneath the explosion and propagates as a wave train past the structure. Airblast-induced ground shock is generated when the traveling overpressure pulse loads the ground surface.

2.6 Special Nuclear Power Plant Loadings

In providing for reliable designs of safety-class nuclear power plant structures, a number of special load cases due to various stipulated accident conditions have to be considered and designed for. External and internally generated missiles have already been mentioned. In the case of high energy pipe rupture, the loading of pipe impact on bumpers, snubbers and supports is to be considered as well as the force due to jet impingement, both of which are highly dynamic phenomena. In boiling water reactors, the sudden activation of safety relief valves gives rise to very high pulsating pressures both inside the pressure suppression pool as well as in the containment structure.

2.7 Mechanical Vibrations

Reinforced concrete structures are often designed to house and/or support mechanical equipment or machinery which exerts mostly steady-state harmonic vibrations to the supporting structure. An important and common example are the support frames or pedestals for turbines. The analysis and design for such loads are more straightforward than for the previously discussed dynamic loads, because they are neither as destructive as blast or missile impact, nor are they as unpredictable as earthquakes, wind, or ocean waves. The operating frequencies are usually known to a high degree of certainty, and by carefully tuning the support structure with the operating characteristics of the equipment, resonance amplification in the important natural modes of vibration can be avoided. Since the response to the operating loads is generally linear, advanced numerical techniques such as the finite element method are seldom required to analyze such structures.

3. MATERIAL PROPERTIES

For the dynamic analysis of concrete structures it is important to determine to what extent the material properties are strain-rate dependent. Most laboratory experiments that have been undertaken to investigate the effect of strain rates on the various properties of concrete were conducted at rates similar to those associated with earthquakes, wind and wave loadings. Test results for strain rates of blast or missile impact loading are much sparser [3,4].

The strain-rate effects on the constitutive relations of concrete have been summarized by Taylor [5]. Experiments with the primary objective of studying the strain-rate effect of earthquake loadings have been conducted by Mahin and Bertero [6]. From this study the following conclusions were drawn.

a) Displacement rates showed negligible effect on initial stiffness.

b) High strain rates of the order 0.05 in./in./sec increased the yield strength of reinforced concrete members by more than 20%, but only in the initial excursion into the inelastic range.

c) In subsequent cycles with loads up to the same displacement amplitudes such as in steady-state cycling, differences in either stiffness or strength were rather small.

d) Strain-rate effects on strength diminished with increased excursions into the strain hardening range.

e) No substantial changes were observed in ductility and overall energy absorption capacity.

Otani [7] has observed that strain rates during oscillations are highest at low stress levels and decrease gradually toward a peak strain. Also, cracking and yielding of a reinforced concrete member reduce the stiffness, hence the period of oscillation becomes longer as structural damage increases. Furthermore, such damage is normally caused by a few lower modes of vibration, whose periods are relatively long. For these reasons, the strain-rate effect may not be as important as some material tests under extraordinary high and mostly constant strain rates indicate. Hysteresis loops obtained from dynamic tests of single-story one-bay reinforced concrete frames compared favorably with those obtained from quasi-static tests [8].

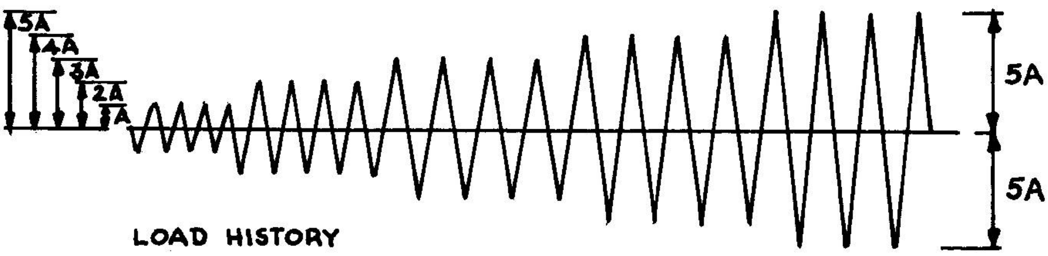
4. REINFORCED CONCRETE BEHAVIOR UNDER CYCLIC LOADING

Research in earthquake engineering and concern for the safety of nuclear power plant facilities have been responsible for most of the current knowledge regarding the mechanical behavior of reinforced concrete under static as well as dynamic loads. The findings of these studies can readily be applied to other kinds of dynamic loads as long as the strain rates are of the same order of magnitude. This is the case for both wind and ocean wave loadings. Their different dynamic characteristics have no immediate bearing upon the material properties.

The awesome complexity of reinforced concrete behavior under cyclic loading makes it all but impossible to derive a generally valid mathematical model that can completely include the effects of all parameters involved. Therefore it is advantageous to isolate flexural and shear behavior from bar slip and bond characteristics as well as experimental setups would permit.

4.1 Flexural Behavior

A number of investigators have studied the flexural behavior of reinforced concrete members under cyclic loads in the laboratory [9,10]. Figure 1 shows a typical moment-curvature diagram obtained from a test of a simply supported beam [10]. As can be noticed, the stiffness decreases gradually as a function of load, and under load reversal forms a pronounced hysteresis loop which provides for an appreciable amount of energy absorption. The hysteresis loops remain remarkably stable for a number of load reversals, provided the maximum prior displacement amplitude is not exceeded. The energy which is input into the structure by, for example, an earthquake, can in this case be effectively dissipated without further deterioration or reduction of ultimate strength. A necessary prerequisite for this desirable behavior is a sufficient amount of confinement reinforcement, as it is required by Appendix A of the ACI Building Code [11], or by special design practice [12].



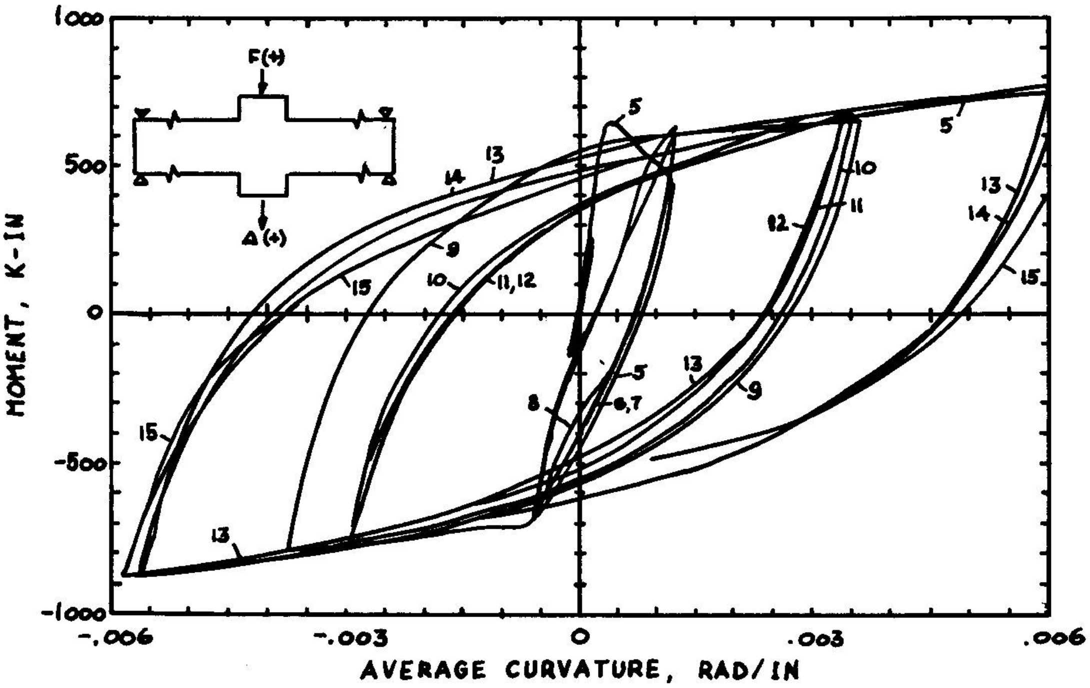


FIG. 1 MOMENT-CURVATURE DIAGRAM (REF. 10)

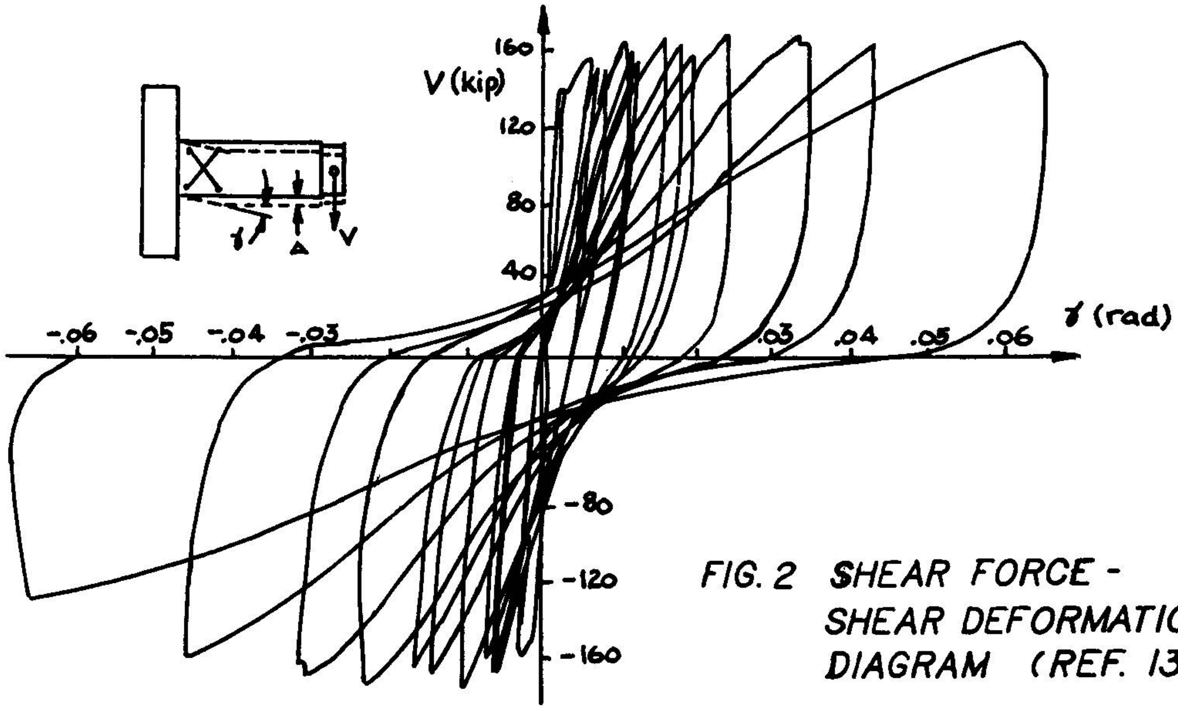


FIG. 2 SHEAR FORCE - SHEAR DEFORMATION DIAGRAM (REF. 13)

In columns, the axial compression tends to decrease the flexural ductility, while delaying the tensile and shear cracking of concrete as well as the yielding of the main reinforcement. The ductility reduction is of most concern to designers; therefore it is good frame design practice to avoid the formation of plastic hinges in columns and to restrict hysteretic energy absorption to the girders.

4.2 Shear Behavior

In order to isolate shear behavior from flexural behavior, relatively short cantilever beams have been tested in the laboratory, and the shear deformations were measured by strain gages arranged in two diagonal and orthogonal directions [13,14]. A typical shear force-shear deformation diagram as shown in Fig. 2 is to some degree reminiscent of the moment-curvature diagram of Fig. 1 [13]. However, it is not an easy matter to completely separate the shear distortions from the other deformation components.

The limited number of experimental data show that in contrast to the flexural stiffness, the shear stiffness typically experiences a gradual increase during loading in either direction, up to an equivalent "yield" level. This behavior gives the shear force-shear deformation hysteresis diagram a "pinched" shape and can be explained as follows. The first complete load cycle far into the inelastic range causes two orthogonal diagonal cracks which reduce the shear stiffness to almost zero for small loads. As the loads in subsequent cycles are increased in either direction, relative slip occurs across the crack until aggregate interlock is activated, which accounts for the sharp increase in apparent shear stiffness.

In contrast to the stable flexural hysteresis behavior, the shear hysteresis loops decay with the number of load reversals, even if the maximum load level is not exceeded. This deterioration is the result of gradual abrasion and cracking of the concrete in the crack region and reduces the energy absorption capacity considerably. This effect of load history on the shear deformation behavior is therefore very important and cannot be neglected when analyzing structures in which the shear behavior has a marked influence on the overall structural response, such as in shear walls, short spandrel beams in coupled shear walls, or in the critical region around the columns in flat plate and flat slab construction.

The shear deformation effect can be modeled in dynamic finite element analysis by either discrete or distributed models. In discrete models, dowel action and aggregate interlock are incorporated by means of special spring elements whose characteristics are determined by experiments. In distributed models, the shear modulus for the elasticity matrix is continually updated in order to simulate the shear behavior.

4.3 Bar Slip and Bond Deterioration

In reinforced concrete frames, beams and columns are normally built integrally so that a certain amount of interaction between adjoining members can be expected. In a typical beam-column assembly such as shown in Fig. 3 [14], for the indicated loading, the right beam moment tends to pull the upper reinforcing bars out of the joint, while the left beam moment helps by pushing. For the bottom reinforcement, both moments are additive in the tendency of pulling the bars to the left. Any kind of cracking in this highly stressed joint region, be it due to flexural or shear action, will only exacerbate the bond deterioration to be expected. The general shape of the moment-bar slip rotation curve is somewhat similar to that of the shear force-shear distortion diagram of Fig. 2, characertized by pinched hysteresis loops. Once the bars are free to move within the joint zone, the contribution of bar slip to the structure displacements can be significant, especially for stiff members with relatively small flexural and shear deformations.

5. MATHEMATICAL MODELS FOR CONCRETE MEMBERS

It is convenient to classify mathematical models for reinforced concrete members into four categories, according to their degree of complexity and accuracy. The most complex and expensive models have been devised for finite element analysis. These allow quite satisfactory simulation of test results for many different situations [15,16]. The large size of the problems in terms of degrees of freedom and the resulting expense preclude the use of finite element models for general design purposes. Only in exceptional cases such as certain safety-class nucler power plant structures is it conceivable that finite element analyses are directly used for design purposes. In the majority of situations, finite element studies may be used to study the behavior of concrete members and to derive from these studies simplified models for more common design applications. Some of the finite element models that have been presented in the literature will be described briefly below.

On the next lower level of sophistication, the moment-curvature relationships of sections are determined by subdividing the cross-sections into a finite number of layers so that these may be called "semi-finite element" models. The computational effort required to establish complete moment-curvature diagrams for each member in a structure is still so overwhelming that general use for structural analysis is seldom feasible.

In order to reach a balance between computational expense and required accuracy for design purposes, another step of simplification has to be taken which leads to the determination of moment-curvature relationships for an entire member directly. Such models may therefore be termed "member-size" models. They can be used directly to determine the member stiffness properties which are in general time- and displacement-dependent. In order to be accurate, the model should reflect the hysteretic characteristics of the member's flexural, shear and bond behavior well into the inelastic region, if the analysis objective includes strength predictions for the structure under consideration. Several such models have been successfully used in the past and will be included in the discussion below.

Models of the lowest level of complexity can be devised to lump the behavioral characteristics of all beams and columns of one building story together and to replace them by an equivalent beam with properties which somehow approximate the mechanical behavior of the structure. Such models are generally too inaccurate and inadequate for design purposes and will not be covered in this report. They may prove useful for certain kinds of studies in which the relative importance of certain parameters is investigated. The comparably small computational effort permits large ranges for such parameters to be studied.

5.1 Finite Element Models

The generality of the finite element method permits us to directly model discontinuities with the mesh layout; concrete elements and steel elements can be modeled separately, with specialized bond elements simulating the interactive forces. All attention can thereafter be concentrated on the establishment of constitutive material laws which are all by themselves complicated indeed.

One model for finite element analysis is based on so-called equivalent uniaxial strains [17,18]. Consider the incremental constitutive equations for an orthotropic plane stress material

$$\begin{Bmatrix} d\sigma_1 \\ d\sigma_2 \\ d\tau_{12} \end{Bmatrix} = \frac{1}{1-\nu_1\nu_2} \begin{bmatrix} E_1 & \nu_2 E_1 & 0 \\ \nu_1 E_2 & E_2 & 0 \\ 0 & 0 & (1-\nu_1\nu_2)G \end{bmatrix} \begin{Bmatrix} d\varepsilon_1 \\ d\varepsilon_2 \\ d\gamma_{12} \end{Bmatrix} \tag{1}$$

where $\nu_1 E_2 = \nu_2 E_1$ and subscripts 1 and 2 denote the current principal stress axes. With the introduction of an equivalent Poisson's ratio, ν,

$$\nu^2 = \nu_1\nu_2 \tag{2}$$

and the assumption that the shear modulus G is approximately independent of axis orientation, i.e.,

$$(1-\nu^2)\,G = \frac{1}{4}\,(E_1 + E_2 - 2\nu\sqrt{E_1E_2}\,) \tag{3}$$

Eq. 1 takes on the form

$$\begin{Bmatrix} d\sigma_1 \\ d\sigma_2 \\ d\tau_{12} \end{Bmatrix} = \frac{1}{1-\nu^2} \begin{bmatrix} E_1 & \nu\sqrt{E_1E_2} & 0 \\ & E_2 & 0 \\ (\text{symm.}) & & \frac{1}{4}\,(E_1+E_2-2\nu\sqrt{E_1E_2}\,) \end{bmatrix} \begin{Bmatrix} d\varepsilon_1 \\ d\varepsilon_2 \\ d\gamma_{12} \end{Bmatrix} \tag{4}$$

Defining incremental equivalent uniaxial strains as

$$\begin{aligned} d\varepsilon_{1u} &= \frac{1}{1-\nu^2}\,[d\varepsilon_1 + \nu\,\frac{E_2}{E_1}\,d\varepsilon_2] \\ d\varepsilon_{2u} &= \frac{1}{1-\nu^2}\,[d\varepsilon_2 + \nu\,\frac{E_1}{E_2}\,d\varepsilon_1] \end{aligned} \tag{5}$$

Eq. 4 simplifies further to the form

$$\begin{aligned} d\sigma_1 &= E_1 d\varepsilon_{1u} \\ d\sigma_2 &= E_2 d\varepsilon_{2u} \\ d\tau_{12} &= G\,d\gamma_{12} \end{aligned} \tag{6}$$

which are of the same form as those for uniaxial stress conditions. The integration of Eq. 6 leads to

$$\varepsilon_{iu} = \int \frac{d\sigma_i}{E_i} \qquad i = 1,2 \tag{7}$$

or its discrete equivalent,

$$\varepsilon_{iu} = \sum_{\text{all load increments}} \frac{\Delta\sigma_i}{E_i} \qquad i = 1,2 \tag{8}$$

where $\Delta\sigma_i$ is the incremental change in the principal stress σ_i.

It should be noted that the ε_{iu} are not real strains; they do not transform under coordinate transformations as real strains do. Further, they are accumulated in the principal stress directions which in general change during loading, so that ε_{1u} for example does not provide a deformation history in a fixed direction, but rather in the continuously changing direction corresponding to principal stress σ_1. Nevertheless, these quantities make it possible to use quite realistic hysteresis rules for plain concrete. The basic idea is that once the stress-strain law has been written in a form similar to that for the uniaxial case, stress-strain curves similar to the uniaxial stress-strain response can be used.

Another, quite different model is based on endochronic theory [19,20] and has been shown to describe very well the response of plain concrete under cyclic loads and also under multiaxial stress. It consists of characterizing the inelastic strain accumulation by a scalar intrinsic time parameter whose increment is a function of strain increments. Thus the form of the endochronic constitutive law for short time deformations of concrete is independent of time,

$$de_{ij} = \frac{ds_{ij}}{2G} + de''_{ij}, \qquad de''_{ij} = \frac{s_{ij}}{2G}\frac{d\zeta}{Z_1}, \qquad d\varepsilon = \frac{d\sigma}{3K} + d\lambda \tag{9}$$

in which $e_{ij} = \varepsilon_{ij} - \delta_{ij}\varepsilon$ are the deviatoric components of the strain tensor ε_{ij}; $\varepsilon = \frac{1}{3}\varepsilon_{kk}$ = volumetric strain; δ_{ij} = Kronecker delta; $s_{ij} = \sigma_{ij} - \delta_{ij}\sigma$ = deviatoric stress components of stress tensor σ_{ij}; $\sigma = \frac{1}{3}\sigma_{kk}$ = volumetric stress. Subscripts i and j refer to Cartesian coordinates x_i, $i = 1,2,3$; K and G are the bulk and shear modulus; e''_{ij} = inelastic deviator strain; λ = inelastic dilatancy; Z_1 = constant; and ζ is a damage measure which is defined by

$$d\zeta = \frac{d\eta}{\left(1 + \dfrac{\beta_1\eta + \beta_2\eta^2}{1 + a_7F_1}\right)F_2} \quad ; \qquad F_2 = 1 + \frac{a_8}{\left(1 + \dfrac{a_9}{\eta^2}\right)J_2(\underset{\sim}{\varepsilon})} \tag{10}$$

$$d\eta = \left\{\frac{a_o}{1 - [a_6I_3(\underset{\sim}{\sigma})]^{1/3}} + F_1\right\}d\xi \; ; \qquad d\xi = \frac{1}{2}\, de_{ij}de_{ij} \tag{11}$$

$$F_1 = \frac{a_2[1 + a_5I_2(\underset{\sim}{\sigma})]\sqrt{J_2(\underset{\sim}{\varepsilon})}}{\{1 - a_1I_1(\underset{\sim}{\sigma}) - [a_3I_3(\underset{\sim}{\sigma})]^{1/3}\}[1 + a_4I_2(\underset{\sim}{\sigma})\sqrt{J_2(\underset{\sim}{\varepsilon})}]} \tag{12}$$

The inelastic dilatancy variable λ is defined by

$$d\lambda = \frac{c_o}{1 - c_1I_1(\underset{\sim}{\sigma})}\left(1 - \frac{\lambda}{\lambda_o}\right)\left\{\left(\frac{\lambda}{\lambda_o}\right)^2 + \left[\frac{J_2(\underset{\sim}{\varepsilon})}{c_2^2 + J_2(\underset{\sim}{\varepsilon})}\right]^3\right\}d\xi \tag{13}$$

and

$$K = \frac{E_o}{3(1-2\nu)}\left(1 - \frac{\lambda}{4\lambda_o}\right) \; ; \qquad G = \frac{E_o}{2(1+\nu)}\left(1 - \frac{\lambda}{4\lambda_o}\right) \tag{14}$$

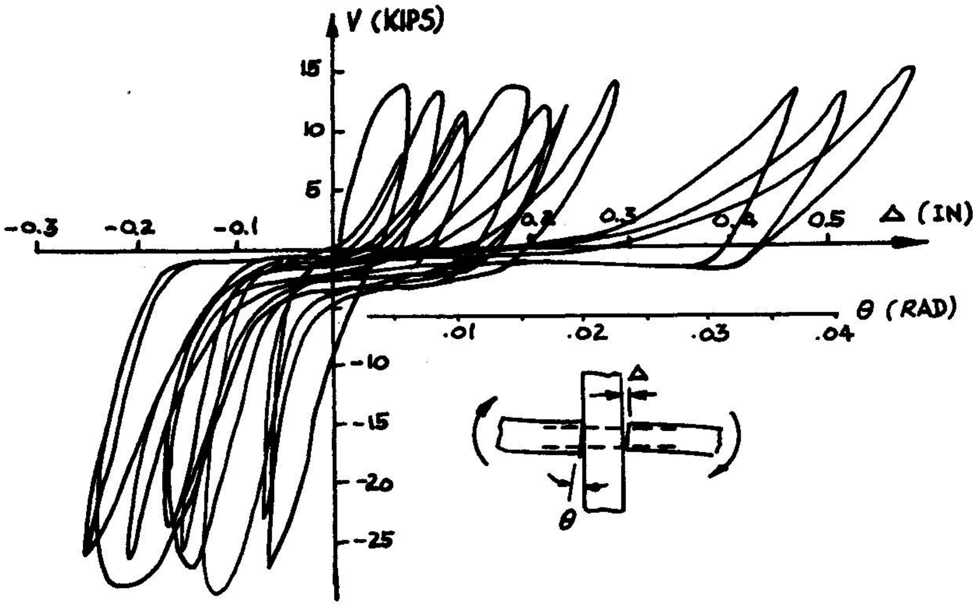


FIG. 3 SHEAR FORCE - BAR SLIP DIAGRAM (REF. 14)

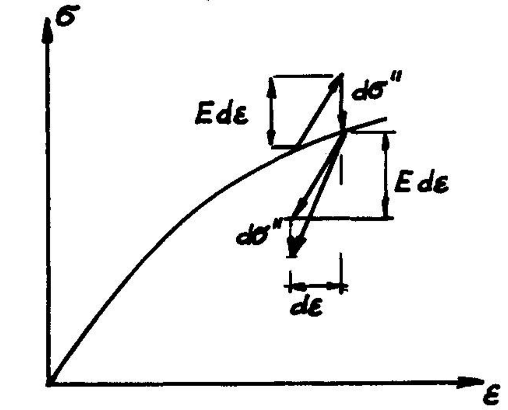


FIG. 4 STRESS-STRAIN LAW ENDOCHRONIC THEORY

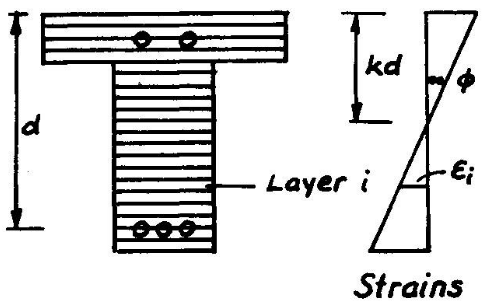


FIG. 5 ELEMENT LAYERS FOR T-SECTION

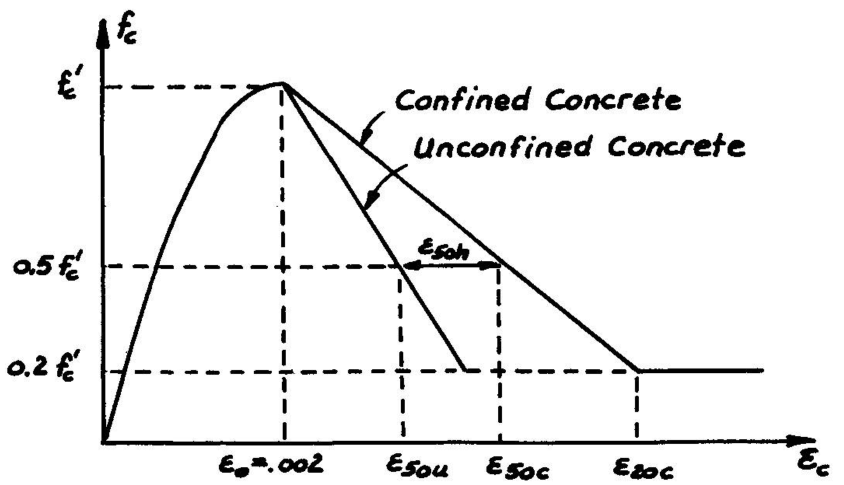


FIG. 6 CONCRETE STRESS-STRAIN LAW (REF. 25)

I_1, I_2, I_3 are the three invariants of the stress tensor, and J_2 is the second invariant of the deviator. The various material parameters can be determined by fitting the constitutive law to test data, such that normal-weight concretes are approximated remarkably well. The above formulation is fully continuous in that no inequalities are needed to distinguish between loading and unloading and various ranges of strain, Fig. 4. Bazant and Bhat claim that the endochronic theory is particularly effective in modeling cyclic responses, inelastic dilatancy due to large shear strains, strain-softening properties, and the hydrostatic pressure effect on triaxial behavior. The theory has been shown to match well the experimental uniaxial, biaxial, and triaxial stress-strain diagrams, including strain softening and failure envelopes, torsion-compression tests, lateral stresses, volume change, unloading and reloading diagrams, and cyclic loadings.

For numerical analysis it is more convenient to recast Eq. 9 in a matrix form, which reads

$$\underset{\sim}{\Delta\sigma} + \underset{\sim}{\Delta\sigma''} = \underset{\sim}{D\Delta\varepsilon}$$

or

$$\begin{Bmatrix} \Delta\sigma_{11} + \Delta\sigma''_{11} \\ \Delta\sigma_{22} + \Delta\sigma''_{22} \\ \Delta\sigma_{33} + \Delta\sigma''_{33} \\ \Delta\sigma_{12} + \Delta\sigma''_{12} \\ \Delta\sigma_{23} + \Delta\sigma''_{23} \\ \Delta\sigma_{31} + \Delta\sigma''_{31} \end{Bmatrix} = \begin{bmatrix} D_{11} & D_{12} & D_{13} & 0 & 0 & 0 \\ & D_{22} & D_{23} & 0 & 0 & 0 \\ & & D_{33} & 0 & 0 & 0 \\ & \text{(symm)} & & D_{44} & 0 & 0 \\ & & & & D_{55} & 0 \\ & & & & & D_{66} \end{bmatrix} \begin{Bmatrix} \Delta\varepsilon_{11} \\ \Delta\varepsilon_{22} \\ \Delta\varepsilon_{33} \\ \Delta\varepsilon_{12} \\ \Delta\varepsilon_{23} \\ \Delta\varepsilon_{31} \end{Bmatrix} \qquad (15)$$

in which $\Delta\sigma''_{ij} = 2G\Delta e_{ij} + 3K\Delta\lambda\delta_{ij}$; δ_{ij} is the Kronecker delta, and $D_{11} = D_{22} = D_{33} = K + 4G/3$; $D_{12} = D_{13} = D_{23} = K - 2G/3$; $D_{44} = D_{55} = D_{66} = 2G$.

Using this material model, the authors were able to correctly reproduce the response of reinforced concrete members under cyclic loading, without the need to adjust any values of the material parameters.

A number of authors have worked with elasto-plastic models which have the advantage of simplicity and computational efficiency [21,22,23]. However, they do not readily allow for effective incorporaton of stiffness degradation under load reversals and increased inelastic deformation and therefore may prove to be inadequate for a number of important applications.

5.2 Semi-Finite Element Models

Park, Kent and Sampson [24] have computed moment-curvature relationships for cyclic loading by subdividing the member section into a number of discrete layers, Fig. 5, with an assumed strain distribution as shown. Stresses in the concrete and steel of the various layers are determined directly from the assumed stress-strain law. With these stresses and the steel and concrete areas, the forces in each layer are easily computed. An iterative technique is then used to correct the assumed strain distribution, or more specifically, the neutral axis position, in order to satisfy the requirement that the sum of all forces acting on the section must be equal to the applied axial load.

The uniaxial stress-strain curve for concrete under monotonic load proposed by Kent and Park [25], Fig. 6, includes the effect of confinement which the authors claim has a major effect only on the unloading branch. The stress-strain curve for the steel reinforcement is modeled by using a Ramberg-Osgood formulation,

$$\varepsilon_s = \frac{f_s}{E_s}\left(1 + \left|\frac{f_s}{f_{ch}}\right|^{r-1}\right) \tag{16}$$

where f_{ch} and r are characteristic parameters chosen to fit experimental data. Using this formulation and a specific set of rules for unloading and reloading, the authors have achieved moderate success in reproducing moment-curvature as well as load-deflection diagrams of cyclically loaded beams and columns.

Ma, Bertero and Popov [26] have also adopted a Ramberg-Osgood formulation for the hysteresis model for reinforcing steel and have succeeded remarkably in reproducing cyclic test results. They also used the monotonic compressive stress-strain curve of Park and Kent as an envelope for cyclic concrete loading. Their improvements over the previous model lie primarily in the detailed rules for unloading and reloading, while the monotonic stress-strain curves for both steel and concrete are essentially the same. As a result, the agreement between theoretical and experimental moment-curvature diagrams is excellent.

Recently, Stanton and McNiven [27] have used system identification techniques to improve and extend Ma's work to include also a bond-slip model. With this work, semi-finite element models have reached a state of refinement where they offer promising potential for practical use.

5.3 Member-Size Models

Neither the finite element nor semi-finite element models are yet of much practical value because of the excessive effort needed to compute the moment-curvature relationship of a section. Realistic concrete structures consist of tens or hundreds of members for which stiffness properties have to be established and updated for each small time or load increment. In order to make a nonlinear analysis feasible, the model for concrete members will have to be simplified considerably. Thus, there is basically no choice but to rely on member-size models capable of describing the end force-end displacement or end moment-end rotation relationships with adequate accuracy, yet without a prohibitive amount of computational effort.

The first such model was the bilinear model proposed by Clough [28]. In its simple form this model cannot simulate the permanent stiffness degradation accompanying the progressive cracking of concrete during large inelastic excursions. In order to improve the model in this regard, Hidalgo and Clough [29] have introduced a stiffness degradation parameter with which the effective concrete modulus of elasticity is decreased uniformly throughout the structure. Since such a parameter has to be related to a maximum deformation response quantity, the maximum amplitude of the first generalized displacement or modal displacement was selected. Assuming that the modal matrix of the structure remains essentially unchanged even during large inelastic deformations, the nodal displacement vector $\{v\}$ can be transformed to modal or generalized coordinates,

$$\{Y\} = [\Phi]^T\{v\} \tag{17}$$

The value of Y_1 at which the first plastic hinge develops in the structure is defined to be the critical threshold beyond which the structure stiffness is effectively reduced.

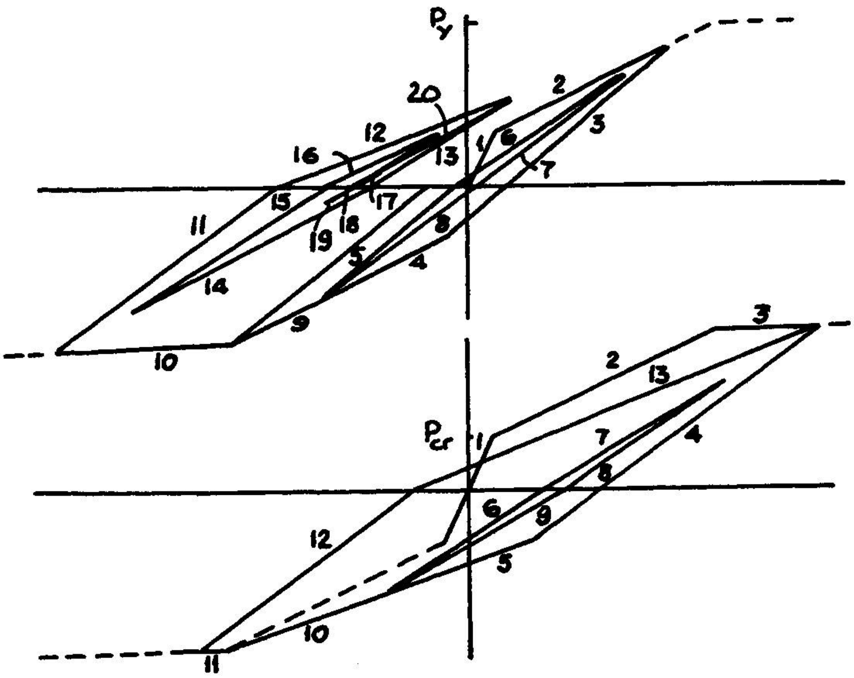


FIG. 7 EXAMPLES OF TAKEDA'S HYSTERESIS RULES (REF. 30)

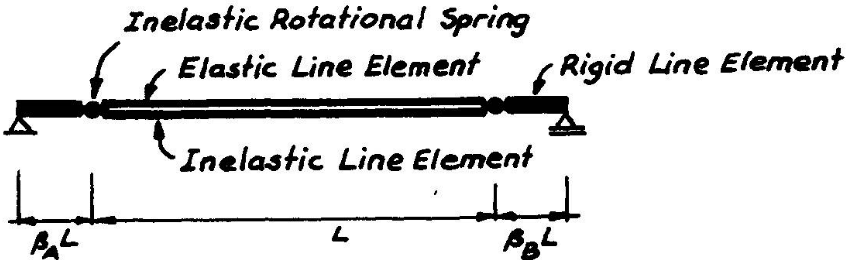


FIG. 8 OTANI'S MODEL (REF. 31)

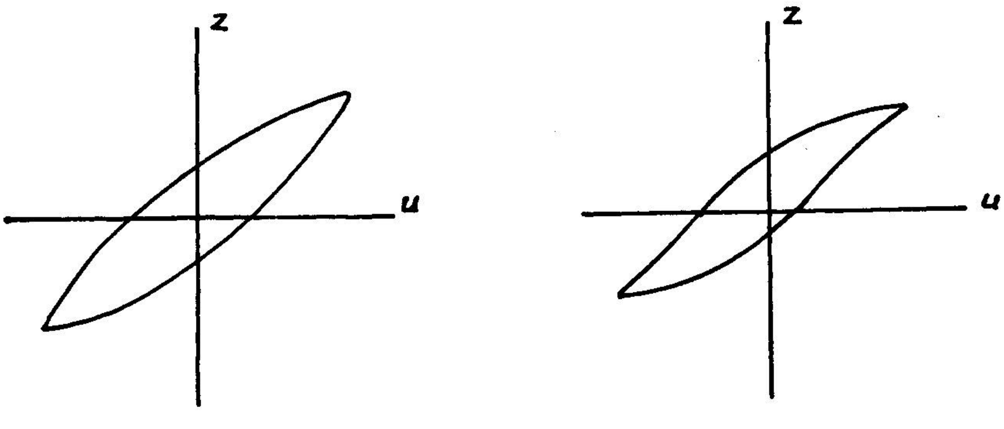


FIG. 9 EXAMPLES OF BABER AND WEN'S MODEL (REF. 32)

This study was rather specialized in that the model parameters were calibrated against test results for a simple two-story one-bay frame, but excellent agreement between experimental and analytical response histories was obtained. A trilinear model was proposed by Takeda, Sozen and Nielsen [30], Fig. 7, in conjunction with an elaborate set of hysteresis rules controlling the response under load reversals.

A somewhat different approach was followed by Otani [31] who added to the conventional bilinear or two-component model concentrated nonlinear springs, Fig. 8, in the vicinity of expected plastic hinges, i.e., near the clear span ends of a member. The flexibility constant of the spring can be adjusted to simulate bond deterioration of the reinforcing steel in the joint core as a function of the steel surface condition and stress history, and the degree of confinement and quality of the concrete. The idealized moment-curvature relationships for sections obtained from the assumed material laws led to results that compared reasonably well with test results for a three-story frame. However, the stiffness degradation mechanism, a simplified version of Takeda's model, tended to underestimate the response in the later phases of the time histories, during periods of reduced ground accelerations.

An interesting model was proposed recently by Baber and Wen [32], wherein four functions are combined to obtain a closed hysteresis loop with sufficient free parameters to model both softening and hardening materials, Fig. 9. The deterioration parameter was chosen to be a function of the total energy dissipated by hysteretic action, although according to some tests [10,13], rather stable hysteresis loops can be obtained, i.e., much energy dissipated without significant stiffness degradation. But the authors' primary objective was not so much to develop a model to accurately reproduce reinforced concrete behavior but rather to permit with relatively little expense the conduct of Monte Carlo type random vibration analyses.

The state of the art in modeling reinforced concrete behavior under cyclic loading has not yet progressed to the point where it is possible to reproduce experimental response data with a cost-effective member-size model that can account for both permanent stiffness degradation in flexural and shear behavior as well as bond deterioration in joint zones, However, it should be stressed that "perfect" match between analytical and experimental responses is neither possible nor particularly needed, because the dynamic response of concrete structures is associated with numerous uncertainties regarding the actual in-place material properties, geometric tolerances, and most significantly of all, the expected load history. The primary goal of mathematical modeling therefore consists rather of capturing the essential mechanical response characteristics within reasonable accuracy. Any remaining discrepancies are justifiably lumped together with the other probabilistic influence factors for assessing the safety or reliability of a complete reinforced concrete structure.

6. DYNAMIC ANALYSIS METHODS

In many situations it is difficult if not impossible to simulate the effects of dynamic loads acting on a structure by using equivalent static loads alone. In these cases, analysis methods have to be employed which take into account the important dynamic effects. These methods are based on the classical theory of vibrations and structural dynamics. For lumped parameter systems such as those resulting from a finite element discretization, the equations of motion can be written in matrix notation as

$$[M]\{\ddot{x}\} + [C]\{\dot{x}\} + [K]\{x\} = \{P(t)\} \tag{18}$$

where $[M]\{\ddot{x}\}$ are the inertia forces and $[C]\{\dot{x}\}$ the viscous damping forces, proportional to velocity, but with constant damping matrix [C]. The dynamic or time-dependent forces $\{P(t)\}$ will in the most general case include the ground motion effect $-[M]\{\ddot{x}_g\}$. The static restoring forces $[K]\{x\}$ are normally nonlinear. This is particularly true for concrete structures for which the stiffness is not only amplitude- but also time-dependent, $[K] = [K(x,t)]$. The nonlinearity of Eq. 18 precludes the use of solution methods that are based on the principle of linear superposition such as the standard normal mode method. For this reason, direct integration methods are more appropriate. In particular, it is practical to solve the equations of motion in incremental steps of small time intervals Δt,

$$[M]\{\Delta\ddot{x}\} + [C]\{\Delta\dot{x}\} + [K]\{\Delta x\} = \{\Delta P(t)\} \tag{19}$$

The time step size Δt should be chosen small enough to allow an adequate definition of the forcing functions $\{P(t)\}$, which are normally assumed to vary linearly within a time step. The actual integration of Eqs.18 or 19 is performed by using one of the numerical integration algorithms that have been developed in recent years [33]. Unconditionally stable methods such as the Newmark β-method or Wilson θ-method deserve particular attention. These methods of integration readily permit constant updating of the structure stiffness $[K(x,t)]$ as prescribed by the mathematical model for the individual structural members. The formation and factorization of the structure stiffness which is normally very large for typical finite element analyses, requires a considerable amount of computational effort. Ways of reducing this effort are described in the literature [34,35].

In view of the randomness of many dynamic loads to which concrete structures may be subjected, it is helpful to divide analysis methods into two major categories, deterministic and probabilistic methods.

For deterministic methods of analysis, the forcing function P(t) is a precisely defined function which for numerical analysis purposes is normally specified in digitized or tabular form, i.e., sampled at a sufficient number of characteristic points to adequately represent the load. Examples for such deterministic loads are actually recorded time histories of earthquake ground accelerations or wind gusts or blast pressure impulse functions.

If the function P(t) is not given as an explicit time history, but only in terms of statistical parameters such as a power spectral density function and maximum intensity, then probabilistic methods of analysis will prove advantageous. For linear systems, the equations of motion are normally solved in the frequency domain. But if the equations are nonlinear because $[K] = [K(x,t)]$, then such methods become very difficult and are not appropriate for large finite element systems. In such cases it is preferable to use Monte Carlo techniques to artificially generate a number of random time histories with spectral properties and maximum intensity which are compatible with the statistical input requirements. The finite element model is then analyzed for each one of the artificial input forcing functions. In this fashion, an ensemble of sample response histories is generated which can be used as a basis for a statistical evaluation of expected maximum displacements, stress levels, cracking, etc. Such statistical information is of considerably more interest to the designing engineer than response values obtained from single deterministic analyses.

Impulsive loads such as blast and missile impact are so completely different from other loadings as to affect not only the modeling considerations but also the analysis methods. When analyzing missile impact problems by the finite element method, the constitutive laws should include the strain-rate dependency of the material properties. The program should also allow for large displacements, slide lines and rezoning. A slide line permits relative slip to occur between finite elements representing the missile and the concrete target, or between portions of the target to allow modeling of a plug being pushed out ahead of the missile. Rezoning permits the division of the target into changing sets of elements during the course of the calculation. It is required in large deformation problems in Lagrangian formulation because the finite elements follow the material motion and may become excessively distorted. When rezoning occurs, the elements are laid out in a regular array, and the motions and stresses from the old elements are mapped into the new elements.

In certain problems involving missile impact or blast loadings, the dynamic response can be treated as wave propagation phenomenon. This can be analyzed by analytical methods only if the waves and structures can be sufficiently simplified, and the material properties assumed to be linear. Since these assumptions seldom apply to concrete structures, the use of discretiziation methods such as finite element or Lagrangian finite difference methods is necessary. Most large-capacity programs are restricted to continuum elements for which the nonlinear stress-strain properties of reinforced concrete are expressed in terms of continuum variables. Some authors have combined structural and continuum elements for those cases in which discrete structural elements such as beams, plates, or shells are embedded in soil or rock, which is treated as a continuum. Computer programs for such hybrid analyses have size limitations for the mathematical models.

Blast problems involving very high pressures can be addressed by treating the concrete as a fluid whose equations of state express the pressure as a function of relative density and specific internal energy.

7. CONCLUSIONS

Reinforced concrete exhibits an extremely complex behavior under dynamic loads, in particular if load reversals into the inelastic range are involved. This is true in the case of buildings subjected to such loadings as severe earthquake ground motions, missile impact or blast loadings. The advancement of numerical modelling techniques, in particular of finite element technology, has permitted researchers to simulate test results remarkably well, as long as shear degradation and bond deterioration do not play a major part in the response. For most practical purposes, however, these elaborate numerical techniques have primarily only academic value. What is needed are member-size models that not only capture the flexural response of reinforced concrete members adequately, but also include the shear degradation and bond deterioration, and which are simple enough to make the nonlinear analysis of realistic structures feasible.

8. ACKNOWLEDGEMENTS

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