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The Analysis and Design of Steel Structures Subject of Variable Repeated Loading

Analyse et dimensionnement des structures en acier soumises à des charges variables répétées

Analyse und Bemessung von Stahltragwerken unter variabler wiederholter Belastung

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1. Introduction

When a framed structure has to resist well-defined repeated loads, the frequency being such as to eliminate considerations of fatigue, two approaches to design are possible. The structure may be designed using elastic methods so that there is a specified factor of safety against yield or, alternatively, the plastic methods of design may be used whereby the designer chooses an appropriate load factor against collapse.

When the nature of the repeated loading is such that two or more combinations of peak loads may occur in approximately cyclic order, plastic design can become inappropriate due to the danger of either alternating plasticity or incremental collapse taking place at load levels below the plastic collapse load. In such cases, both steel and reinforced concrete structures may be rationally and economically designed on the basis of the shakedown load. This can be seen as a hybrid elastic-plastic approach whereby the structure has a specified reserve of strength against the onset of either alternating plasticity or incremental collapse.

The main objections to such an approach are on the grounds of analytical difficulty. In this paper, a suitable family of computer-orientated techniques for both analysis and design are described. These are not significantly more expensive to use than the usual approaches to automatic plastic analysis and design (which they embrace as a special case). The basis of these techniques is an unusual formulation of the equilibrium equations for plane, rigid-jointed frames.

2. Equilibrium Equations for Plane Frames

These equations have two forms and it is convenient to consider the general case first. It is assumed that the shape of the structure has been determined and that the member sizes are required. The computation is commenced by making an arbitrary assumption regarding the member cross sections (eg all members identical). Then hinges are inserted, one for each degree of statical indeterminacy in such a way that they do not constitute a mechanism.

The complete stiffness equations for the modified structure can be written down with the terms corresponding to the inserted hinges partitioned from the terms corresponding to the original structure.

$$\left[\begin{array}{c|c} S_{11} & S_{12} \\ \hline S_{21} & S_{22} \end{array}\right] \left[\begin{array}{c} D \\ \hline \theta_H \end{array}\right] = \left[\begin{array}{c} L \\ \hline M_H \end{array}\right] \qquad \ldots\ldots \quad (1)$$

Thus, S_{11} is stiffness matrix of the original structure, and the other submatrices contain additional terms associated with the extra hinges. L and D are matrices containing the vectors of applied load and corresponding displacements and M_H and θ_H are matrices containing the moments at and rotations of the hinges. We are only concerned with the terms above the partition line and these can be considered in two alternative ways, as follows.

(1) For the original structure, loaded by several alternative load combinations each of which has a corresponding column in the load matrix L, the rotations θ_H are zero so that the displacements D_L due to L are given by

$$D_L = S_{11}^{-1}\, L \qquad \ldots\ldots \quad (2)$$

(2) For the unloaded structure, consider a unit rotation at the location of each inserted hinge in turn. Then θ_H becomes a unit matrix and L is null so that the corresponding displacements D_θ are given by

$$D_\theta = S_{11}^{-1} \left[-S_{12}\right] \qquad \ldots\ldots \quad (3)$$

It is advantageous to combine (2) and (3) so that both can be solved in one operation.

$$\left[\begin{array}{c|c} D_L & D_\theta \end{array}\right] = S_{11}^{-1} \left[\begin{array}{c|c} L & -S_{12} \end{array}\right] \qquad \ldots\ldots \quad (4)$$

The result on the left hand side of (4) consists of a series of displacement vectors which can be considered one at a time and the distributions of bending moment calculated for the 'c' critical sections of the structure. Thus are obtained:-

(1) From vectors D_L, bending moments $\mathcal{M}_i$ that are in equilibrium with the applied loads L. (These are in fact correct elastic bending moments for an assumed structure).

(2) From vectors D_θ, distributions of residual bending moment K_{ij} that are in equilibrium with zero applied load and are mutually independent. The number of these distributions is equal to the degree or redundancy 'r' of the structure.

Combining these results gives generalised equilibrium equations for the structure in which x_j is an arbitrary multiplier for the jth distribution of residual bending moment.

$$M_i = \mathcal{M}_i + \sum_{j=1}^{r} K_{ij}\, x_j \quad (i = 1,\ 2,\ \ldots\ldots\ c) \qquad \ldots\ldots \quad (5)$$

The cost of these equations is a single elastic analysis of the structure with a possibly large number of right hand sides. The advantage is complete generality regarding structural shape and loading and a convenient form for the techniques that follow.

3. Analysis for the Shakedown Load - A Problem Orientated Technique

The most general form of the equilibrium equations has been derived with design problems in mind. For the analysis of a given structure, equations (1) are the correct stiffness equations and it is convenient to insert a hinge at each critical section in order to obtain the particular form of the equilibrium equations.

$$M_i = \mathcal{M}_i + \sum_{j=1}^{c} K_{ij}\,\theta_j \quad (i = 1, 2, \ldots\ldots c) \qquad \ldots\ldots \quad (6)$$

In (6), θ_j are actual plastic hinge rotations which are initially zero and $\mathcal{M}_i$ are actual elastic bending moments from which can be extracted the maximum and minimum values at each critical section after considering all possible combinations of loading.

The analysis for the shakedown load involves following the formation of plastic hinges as the load level increases while maintaining a distribution of residual bending moment that satisfies the shakedown theorem(1). Let first yield take place at critical section 'ℓ' with load factor λ_1 and bending moments typified by M_1. Then for further loading, the bending moment at section ℓ must remain at the full plastic moment 'M_p' and one or other of (7) must apply.

$$\begin{aligned} &\lambda\mathcal{M}^{max} + K_{\ell\ell}\,\theta_\ell = +M_p \\ \text{or} \quad &\lambda\mathcal{M}^{min} + K_{\ell\ell}\,\theta_\ell = -M_p \end{aligned} \qquad \ldots\ldots \quad (7)$$

At an arbitrary load factor $\lambda_1 + d\lambda$, θ can be evaluated from (7) and bending moments typified by M_1' calculated. A linear prediction can then be made for the load factor λ_2 at which the next hinge forms.

$$\lambda_2 = \lambda_1 + d\lambda\,\frac{(\pm M_p - M_1)}{(M_1' - M_1)} \qquad \ldots\ldots \quad (8)$$

The smallest λ_2 obtained when the maximum and minimum elastic bending moments are considered at each critical section in turn locates the next plastic hinge at (say) section 'm'. Equations (9) now govern further loading.

$$\begin{aligned} &\lambda\mathcal{M}_\ell^{\,max/min} + K_{\ell\ell}\,\theta_\ell + K_{\ell m}\,\theta_m = \pm M_p \\ &\lambda\mathcal{M}_m^{\,max/min} + K_{m\ell}\,\theta_\ell + K_{mm}\,\theta_m = \pm M_p \end{aligned} \qquad \ldots\ldots \quad (9)$$

The plastic hinge rotations θ_ℓ and θ_m can be calculated at an arbitrary load factor greater than λ_2, a linear prediction made for the load factor at which the next plastic hinge forms, and the process continued. The formation of successive plastic hinges can be followed until the shakedown load is reached when an incremental collapse mechanism exists and the matrix of participating influence coefficients for residual bending moment becomes singular.

Further consideration of this technique, including a simple yet accurate allowance for frame instability, is given elsewhere(2).

4. Analysis for the Shakedown Load - A Linear Programming Technique

Here, the equilibrium equations are also obtained from an analysis of the actual structure so that the maximum and minimum elastic bending moments are available but we use the general form of the equations. The static formulation of the problem then arises directly from the shakedown theorem, thus:-

$$\begin{aligned}
&\text{Maximise} \quad \lambda \\
&\text{Subject to} \quad \lambda \mathcal{M}_i^{max} + \sum_{j=1}^{r} K_{ij} x_j - \sum_{j=1}^{r} K_{ij} x'_j \leqslant M_{pi} \\
&\qquad\qquad -\lambda \mathcal{M}_i^{min} - \sum_{j=1}^{r} K_{ij} x_j + \sum_{j=1}^{r} K_{ij} x'_j \geqslant M_{pi} \qquad (i = 1, 2, \ldots\ldots c) \qquad \ldots\ldots \ (10)
\end{aligned}$$

This is a linear programming problem which can be readily solved using the dual simplex algorithm. The alternative kinematic formulation of the problem can be obtained as the linear programming dual of (10) and given physical meaning[3] but the above statement has simpler constraints and for that reason is preferred.

Although the linear programming approach leads to a viable method of analysis, the problem-orientated approach is considerably more efficient. Indeed, the method described in section 3 is believed to be the most efficient method available for the calculation of the failure loads of plane frames with frame instability whether or not repeated loading is involved.

5. Automatic Design for Minimum Weight at the Shakedown Load

Here, the linear programming approach becomes more appropriate but a new difficulty presents itself. The statement of the problem depends on a knowledge of the maximum and minimum elastic bending moments which, in turn, depend on the relative member stiffnesses which are initially unknown. An iterative technique is required whereby assumed member stiffnesses are successively improved and this is found to converge rapidly.

The statement of the problem in its static form involves the usual minimisation of a linear weight function which is the sum of the products of length 'L_k' and full plastic moment 'M_{pk}' taken over the 'n' groups of members of identical full plastic moment. Application of the shakedown theorem results in a static yield condition similar to that obtained in section 4 so that the full statement is:-

$$\begin{aligned}
&\text{Minimise} \quad \sum_{k=1}^{n} L_k M_{pk} \\
&\text{Subject to} \quad M_{pk} - \sum_{j=1}^{r} K_{ij} x_j + \sum_{j=1}^{r} K_{ij} x'_j \geqslant \mathcal{M}_i^{max} \qquad (i = 1, 2, \ldots\ldots c) \\
&\qquad\qquad M_{pk} + \sum_{j=1}^{r} K_{ij} x_j - \sum_{j=1}^{r} K_{ij} x'_j \geqslant \mathcal{M}_i^{min} \qquad \ldots\ldots \ (11)
\end{aligned}$$

This too is a linear programming problem in a form readily solved by the use of the dual Simplex algorithm. A similar statement to this, and its kinematic dual, have been discussed in connection with plastic design and it has been shown that a simple serviceability constraint can be readily incorporated[3].

6. Approximate Minimum Weight Design - A Problem Orientated Approach

This technique results in designs that are almost invariably within 1 or 2% of the minimum weight but at a greatly reduced cost in computer time. The method is best described with reference to an example.

Fig 1 shows a frame which represents a simple design problem in which two unknown full plastic moments are to be chosen, M_{p1} for the columns and M_{p2} for the beam. The vertical load can vary between zero and 1.0. Table 1 summarises the design process. Each of the seven critical sections has two columns, the numbers corresponding with Fig 1. The first number refers to the maximum elastic bending moment, the second to the minimum and these moments are evaluated in arbitrary units for a frame of arbitrary stiffness in row 1. Three residual bending moment distributions corresponding to unit rotations of hinges at sections 2, 3 and 4 are shown in rows 2, 3 and 4. The merit of any particular design is judged by the usual linear weight function, in this case $Z = 200M_{p1} + 100M_{p2}$, which appears in the right hand column.

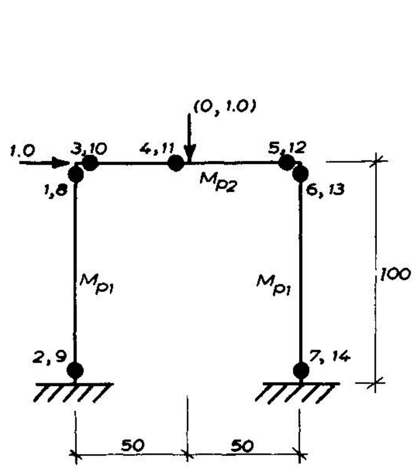

Fig 1. Simple Design Problem

The design process involves successively reducing the value of the weight function while satisfying the condition of static yield; equilibrium being ensured by the use of the generalised equilibrium equations (5).

The numerically largest bending moments present in the columns and the beams represent a feasible initial design. Thus at the commencement, $M_{p1} = 32.74$ units (section 14) and $M_{p2} = 29.76$ units (section 5) and Z = 9542 units. The moments governing the current design at each stage are marked with asterisks.

	Plastic Moment	M_{p1}		M_{p2}			M_{p1}		M_{p1}		M_{p2}			M_{p1}		Z
	Section	1	2	3	4	5	6	7	8	9	10	11	12	13	14	
1		-13.10	-24.41	21.43	0	29.76*	-21.43	-28.57	-21.43	-28.57	13.10	-16.67	21.43	29.76	-32.74*	9524
2	m_2	952	20950	952	3333	7619	-7619	-12380	-952	20950	952	3333	7619	-7619	-12380	
3	m_3	-10950	952	10950	-6667	-2381	2381	7619	-10950	952	10950	-6667	-2381	2381	7619	
4	m_4	6667	3333	-6667	6667	6667	-6667	-3333	6667	3333	-6667	6667	6667	-6667	-3333	
5	$.000568\ m_3$	-6.22	+0.54	+6.22	-3.79	-1.35	+1.35	+4.33	-6.22	+0.54	+6.22	-3.79	-1.35	+1.35	+4.33	
6		-19.32	-23.87	27.65	-3.79	28.41*	-20.07	-24.24	-27.65	-28.03	19.32	-20.46	20.07	-28.41*	-28.41*	8522
7	$m_2' = m_2 + .909\ m_3$	-10910	21820	10910	-2728	5454	-5454	-5454	-10910	21820	10910	-2728	5454	-5454	-5454	
8	$m_4' = m_4 - .636\ m_3$	13640	2727	-13640	10910	8182	-8182	-8182	13640	2727	-13640	10910	8182	-8182	-8182	
9	$-.0000344\ m_4'$	-0.47	-0.09	+0.47	-0.38	-0.28	+0.28	+0.28	-0.47	-0.09	+0.47	-0.38	-0.28	+0.28	+0.28	
10		-19.79	-23.96	28.12	-4.17	28.13*	-19.79	-23.96	-28.12*	-28.13*	19.79	-20.83	19.79	-28.13*	28.13*	8438
11	$m_2'' = m_2' - 2.50\ m_4'$	-45000	15000	45000	-30000	-15000	15000	15000	-45000	15000	45000	-30000	-15000	15000	15000	
12	$-(0.52 \times 10^{-7}) m_2''$	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	
13		-19.79	-23.96	28.12*	-4.17	28.12*	-19.79	-23.96	28.12	28.12*	19.79	-20.83	19.79	-28.12*	-28.12*	8437

Table 1. Shakedown Design of Simple Portal Frame

This design can be improved by either adding or subtracting a small proportion of any of the available residual bending moment distributions in rows 2, 3 and 4. The one that makes possible the greatest weight reduction is to be chosen. For example, if a small proportion of row 2 is subtracted from row 1, the moments at sections 5 and 14 both reduce and the weight reduces. A limit is reached when the increasing moment at some other section becomes equal to the falling moment governing the design at that section. When each of rows 2, 3 and 4 are considered in this way, it is found that the greatest weight reduction is obtained by adding 0. 000568 x row 3 to row 1 thus equalising the moments at sections 13 and 14. This step is summarised in rows 5 and 6 in Table 1.

Before proceeding further, it is necessary to ensure that this equality of bending moments is not violated at a later stage. This can be readily achieved by combining the distributions of residual bending moment so that they too exhibit the same equality. This involves reducing by one the number of such distributions, the equations necessary to achieve this and the resulting distributions being shown in rows 7 and 8.

Rows 6 - 8 can now be seen to be a starting point for another weight reducing step, being directly analogous to rows 1 - 4. This step is summarised by rows 9 and 10. The process can be continued until each of the available residual bending moment distributions has been combined with the elastic bending moment distribution at which stage no further weight reducing steps can be made. This final design is given in row 13 of the table.

For this example the final design is in fact optimal, being identical to that obtained by linear programming. In general this will not be so but the final design will be so close to the optimum that the difference is unsignificant when translated into available sections. Here also, iteration of member stiffnesses is usuall required though with this particular example, the final design results in a frame of uniform section which coincides with the initial assumption so that no iteration is required.

This technique has been described in greater detail, with particular reference to collapse design, in reference 4, where examples of its use are given and compared with solutions obtained by linear programming.

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SUMMARY

This paper is concerned with plane steel frameworks for which the limiting load is the shakedown load. It describes techniques for both analysis and design which are all based on an unusual derivation of generalised equilibrium equations. Both linear programming formulations and problem-orientated techniques are described, the latter being considerably more efficient.

RESUME

Ce travail étudie des cadres plans en acier pour lesquels la charge limite est la charge de rupture. Il décrit des méthodes pour l'analyse et le dimensionnement basées sur une dérivation inhabituelle des équations d'équilibre généralisées. Il présente à la fois les méthodes de programmation linéaire et les techniques spécifiques adaptées au problème, les dernières étant plus efficaces.

ZUSAMMENFASSUNG

Dieser Aufsatz befasst sich mit ebenen Stahlrahmen, für die die Grenzlast die "Shakedown"-Last ist. Er beschreibt Methoden zur Berechnung und Bemessung, die sich alle auf einer ungewöhnlichen Ableitung der verallgemeinerten Gleichgewichtsgleichungen aufbauen. Es werden Formulierungen zum linearen Programmieren und problemorientierte Techniken beschrieben, wobei letztere bedeutend leistungsfähiger sind.

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