The analysis of thin, thick and sandwich plates by finite strip method

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The Analysis of Thin, Thick and Sandwich plates by the Finite Strip Method

Analyse de plaques minces, épaisses et sandwich par la méthode des bandes finies

Berechnung dünner, dicker und Sandwichplatten mittels der finiten Streifenmethode

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Introduction

The use of the finite strip method for analysing elastic plates is well established [1–3]. The method, which is similar in principle to the finite element technique, assumes the plate to be an assemblage of narrow longitudinal strips and defines the displacement field in terms of one-way slab functions across the width of the strip and basic series function in the longitudinal direction.

Previous formulations of the finite strip method for the analysis of plate bending have invariably used slab functions which constrain the plate to obey the Kirchoff's normality hypothesis. Consequently, no allowance is made for the effects of transverse shear deformations. In thick and andwich plate situations the influence of transverse shear on the deformations and stresses is quite significant and cannot be neglected in the analysis.

In this paper finite strip formulations are presented for the elastic analysis of rectangular and curved plates with opposite simply supported ends. The formulations involve transverse shear deformation which is included in the analysis by discarding the Kirchoff's normality hypothesis and specifying independently the transverse displacement and normal rotations of the plate.

Finite Strip Formulation of Rectangular Plates

A detailed description of the ingredients required for the implementation of the finite strip method in the analysis of plate bending has been given by Cheung [1]. The essential steps involved in deriving the stiffness and load matrices of a rectangular finite strip in which the effects of transverse shear deformation are considered, are now given.

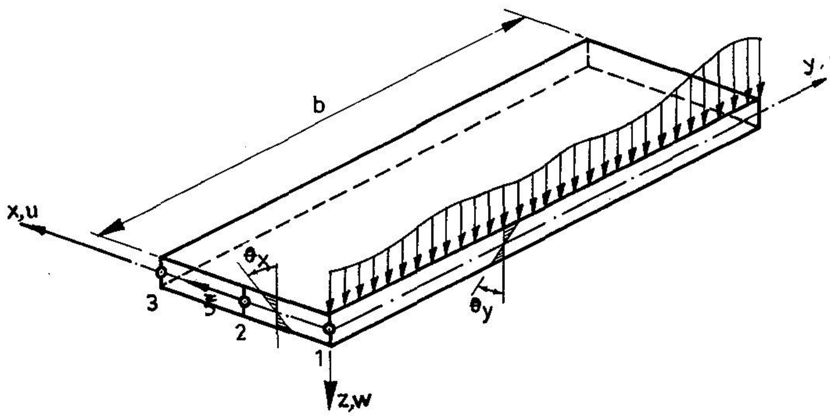

Fig. 1. Rectangular finite strip.

Figure 1 shows a typical finite strip with three nodal lines. The deformation of the plate is defined in terms of the transverse displacement w, and the rotations θ_x and θ_y of the normal to the reference xy-plane. The variables w, θ_x and θ_y are independently specified so that plate normals are not constrained to remain normal during deformation as in the classical thin plate theory. This permits the plate to experience transverse shear deformations although the transverse cross-sections of the plate do not warp out of their plane during deformation.

The displacement vector at any point (x, y, z) of a simply supported strip can then be written in series form as [4]

$$\begin{aligned} u &= -z \sum_{l=1}^{\infty} \sum_{i=1}^{n} N_i \, \theta_{xi}^l \sin \frac{l\pi y}{b} \\ v &= -z \sum_{l=1}^{\infty} \sum_{i=1}^{n} N_i \, \theta_{yi}^l \cos \frac{l\pi y}{b} \\ w &= \sum_{l=1}^{\infty} \sum_{i=1}^{n} N_i \, w_i^l \sin \frac{l\pi y}{b} \end{aligned} \tag{1}$$

where n denotes the number of nodal lines per strip; and the vector of the nodal-line displacement amplitudes is prescribed for the l^{th} harmonic as

$$\{\delta_i^l\} = \{w_i^l, \; \theta_{xi}^l, \; \theta_{yi}^l\}^T \tag{2}$$

The shape functions N_i are simple Lagrangian interpolation functions corresponding to those of an n-noded beam element. In this paper only parabolic shape functions will be considered for which

$$n = 3, \; N_1 = -\tfrac{1}{2}s(1-s), \; N_2 = 1 - s^2 \text{ and } N_3 = \tfrac{1}{2}s(1+s) \tag{3}$$

At $y = 0$ or $y = b$ we shall always have

$$w = u = \frac{\delta v}{\delta y} = 0$$

which corresponds to simply supported boundary conditions.

For orthotropic situations the constitutive relation is given by

$$\begin{Bmatrix} M_x \\ M_y \\ M_{xy} \\ Q_x \\ Q_y \end{Bmatrix} = \begin{bmatrix} D_x & D_1 & 0 & 0 & 0 \\ D_1 & D_y & 0 & 0 & 0 \\ 0 & 0 & D_{xy} & 0 & 0 \\ 0 & 0 & 0 & S_x & 0 \\ 0 & 0 & 0 & 0 & S_y \end{bmatrix} \begin{Bmatrix} -\dfrac{\delta\theta_x}{\delta x} \\ -\dfrac{\delta\theta_y}{\delta y} \\ -\left(\dfrac{\delta\theta_x}{\delta y} + \dfrac{\delta\theta_y}{\delta x}\right) \\ \dfrac{\delta w}{\delta x} - \theta_x \\ \dfrac{\delta w}{\delta y} - \theta_y \end{Bmatrix}$$

or

$$\{M\} = [D] \sum_{l=1}^{\infty} \sum_{i=1}^{n} [B_i^l] \{\delta_i^l\} = [D] \sum_{l=1}^{\infty} [B^l][\delta^l] = [D][B]\{\delta\} \tag{5b}$$

where the strain submatrix $[B_i^l]$ is given by

$$[B_i^l] = [\bar{B}_i^l] \, sin \frac{l\pi y}{b} + [\bar{\bar{B}}_i^l] \cos \frac{l\pi y}{b} \tag{6a}$$

$$\text{with } [\bar{B}_i^l] = \begin{bmatrix} 0 & -\dfrac{\delta N_i}{\delta x} & 0 \\ 0 & 0 & \dfrac{l\pi}{b} N_i \\ 0 & 0 & 0 \\ \dfrac{\delta N_i}{\delta x} & -N_i & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{6b}$$

$$\text{and } [\bar{\bar{B}}_i^l] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \dfrac{l\pi}{b} N_i & -\dfrac{\delta N_i}{\delta x} \\ 0 & 0 & 0 \\ \dfrac{l\pi}{b} N_i & 0 & -N_i \end{bmatrix} \tag{6c}$$

With the strain and property matrices known, the stiffness matrix of the strip can be calculated from the well known relationship [5]

$$[k] = \int [B]^T [D] [B] \, dx \, dy \tag{7a}$$

in which a typical submatrix of $[k]$ linking harmonics l and m is given by

$$[k^{lm}] = \int [B^l]^T [D] [B^m] \, dx \, dy \tag{7b}$$

The integration is carried out over the area of the strip. It is done explicitly in the longitudinal direction but it might be necessary to perform it numerically across the width of the strip. In the examples presented a 2-point Gaussian integration rule has been used for this purpose. It will be noted, however, that for the exact integration of equations (7) a 3-point Gaussian rule is needed; but in accordance with the recommendations of reference [4], a lower 2-point rule has been adopted in order to improve the strip performance and to eliminate the spurious shear effects inherent in this type of formulation. Numerical integration also facilitates the treatment of variable thickness [4].

The loading on the plate must be resolved into basic series function in the longitudinal direction. For instance, a distributed transverse loading of intensity q can be expressed in a series form as

$$q = \sum_{l=1}^{\infty} q^l \sin \frac{l\pi y}{b} \tag{8}$$

The consistent load vector corresponding to this loading can be obtained from the virtual work principle [5] as

$$\{F^l\} = -\int [N_1, 0, 0, N_2, 0, 0, ...]^T q^l \sin^2 \frac{l\pi y}{b} dx\,dy \tag{9}$$

for the l^{th} harmonic.

Extension to Curved Plate Situations

The formulation can easily be extended to deal with curved strips generated by sweeping the section along a circular arc as shown in Figure 2.

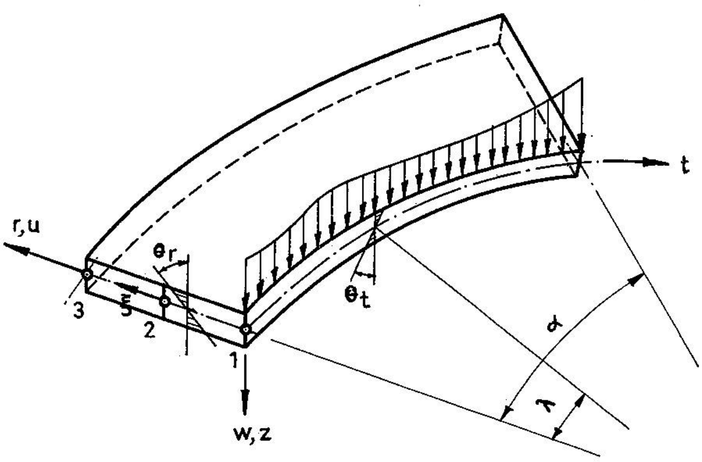

Fig. 2. Curved finite strip.

The variable co-ordinate y is replaced by an angle λ and the span b by an angle α, and the displacements are described by equation (1) now interpreted in polar co-ordinates as follows:

$$
\begin{aligned}
u &= -z \sum_{l=1}^{\infty} \sum_{i=1}^{n} N_i\, \theta_{ri}^{l} \sin \frac{l\pi\lambda}{\alpha} \\
v &= -z \sum_{l=1}^{\infty} \sum_{i=1}^{\infty} N_i\, \theta_{ti}^{l} \cos \frac{l\pi\lambda}{\alpha} \\
w &= \sum_{l=1}^{\infty} \sum_{i=1}^{n} N_i\, w_i^{l} \sin \frac{l\pi\lambda}{\alpha}
\end{aligned}
\tag{10}
$$

The constitutive relationship has also to be changed into polar co-ordinates, and for orthotropic situations it becomes

$$
\begin{Bmatrix} M_r \\ M_t \\ M_{rt} \\ Q_r \\ Q_t \end{Bmatrix} =
\begin{bmatrix} D_r & D_1 & 0 & 0 & 0 \\ D_1 & D_t & 0 & 0 & 0 \\ 0 & 0 & D_{rt} & 0 & 0 \\ 0 & 0 & 0 & S_r & 0 \\ 0 & 0 & 0 & 0 & S_t \end{bmatrix}
\begin{Bmatrix} -\dfrac{\delta\theta_r}{\delta r} \\ -\dfrac{1}{r}\left(\theta r - \dfrac{\delta\theta_t}{\delta\lambda}\right) \\ -\dfrac{1}{r}\left(\dfrac{\delta\theta_r}{\delta\lambda} + r\dfrac{\delta\theta_t}{\delta r} - \theta_t\right) \\ \dfrac{\delta w}{\delta r} - \theta_r \\ \dfrac{1}{r}\dfrac{\delta w}{\delta\lambda} - \theta_t \end{Bmatrix}
\tag{11}
$$

The strain submatrices $[\bar{B}_i^l]$ and $[\bar{\bar{B}}_i^l]$ therefore become

$$
[\bar{B}_i^l] = \begin{bmatrix} 0 & -\dfrac{\delta N_i}{\delta r} & 0 \\ 0 & -\dfrac{N_i}{r} & \dfrac{l\pi}{r\alpha} N_i \\ 0 & 0 & 0 \\ \dfrac{\delta N_i}{\delta r} & -N_i & 0 \\ 0 & 0 & 0 \end{bmatrix}
\tag{12a}
$$

and

$$
[\bar{\bar{B}}_i^l] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -\dfrac{l\pi}{r\alpha} N_i & -\dfrac{\delta N_i}{\delta r} + \dfrac{N_i}{r} \\ 0 & 0 & 0 \\ \dfrac{l\pi}{r\alpha} N_i & 0 & -N_i \end{bmatrix}
\tag{12b}
$$

Numerical Examples

In order to verify the accuracy of the formulation, a uniformly loaded simply supported square sandwich plate having flexural and shear rigidities of D and $\frac{100D}{a^2}$, respectively, is first considered. Because of symmetry only a half of the plate divided into one, two and three strips is analysed, with the strips running parallel to the y-axis. Results of the maximum deflections, shearing forces, bending and twisting moments are given in Table 1, for various numbers of strips at different harmonic terms. It can be seen that the results converge rapidly as both the number of strips and the harmonic terms increase. The solution obtained using 3 strips with 4 harmonics agrees closely with that given by PLANTEMA [6].

Table 2 shows a convergence study involving a thin, isotropic, simply supported square plate also analysed using one, two and three longitudinal strips.

Table 3 shows a study of a uniformly loaded, orthotropic, simply supported square plate analysed using two strips in a symmetric half of the plate with four harmonic terms. Two cases of orthotropy were considered with the stiffness rigidities given in Table 3.

Table 1. Maximum deflection, moments and shearing forces for a square simply supported sandwich plate under uniform loading.

No. of strips	No. of harmonics	Central deflection w_{max}	Central bending moments $(M_x)_{max}$	$(M_y)_{max}$	Twisting moment $(M_{xy})_{max}$	Shearing forces at mid-edges $(Q_x)_{max}$	$(Q_y)_{max}$
1	1	0.00495	0.0577	0.0549	0.0319	0.487	0.242
	2	.00486	.0562	.0503	.0336	.460	.278
	3	.00487	.0564	.0512	.0339	.468	.287
	4	.00487	.0563	.0509	.0341	.464	.291
2	1	0.00489	0.0502	0.0520	0.0309	0.425	0.244
	2	.00479	.0487	.0475	.0326	.387	.288
	3	.00481	.0491	.0485	.0329	.398	.304
	4	.00480	.0489	.0481	.0331	.393	.312
3	1	0.00489	0.0496	0.0518	0.0305	0.401	0.244
	2	.00479	.0481	.0473	.0322	.359	.288
	3	.00480	.0484	.0483	.0326	.373	.304
	4	.00480	.0483	.0479	.0327	.367	.312
Exact solution [6]		0.00480	0.0479		0.0325	0.338	
Multiplier		$\frac{qa^4}{D}$	qa^2		qa^2	qa	

Table 2. Central deflection, moments and edge shears for an isotropic, square, simply supported thin plate under uniform loading

($\mu = 0.3$, $\frac{h}{a} = 0.01$, total number of harmonics terms = 4)

No. of strips	Central deflection	Central bending moments		Shearing forces at mid-edges	
	w_{max}	$(M_x)_{max}$	$(M_y)_{max}$	$(Q_x)_{max}$ [1]	$(Q_y)_{max}$
1	0.00415	0.0563	0.0509	0.315	0.291
2	0.00407	0.0489	0.0481	0.328	0.312
3	0.00407	0.0482	0.0479	0.332	0.312
Thin plate solution [1]	0.00406	0.0479		0.338	
Multiplier	$\frac{qa^4}{D}$	qa^2		qa	

[1] Values interpolated from those obtained at the integration points.

Table 3. Central deflection and moments for an orthotropic, square, simply supported plate under uniform loading.

Source	Central deflection w_{max}		$(M_x)_{max}$		$(M_y)_{max}$	
	A	*B*	*A*	*B*	*A*	*B*
Present solution	0.00153	0.000633	0.0178	0.00812	0.0774	0.0991
Ref. [1]	0.00153	0.000633	0.177	0.00833	0.0777	0.0995
Exact solution [7]	0.00152	0.000633	0.178	0.00838	0.774	0.0993
Multiplier	$\frac{qa^4}{D_x}$		qa^2		qa^2	

Elastic properties
Case A: $D_y = 5.0625D_x, D_1 = 0.375D_x, D_{xy} = 0.9375D_x, S_x = S_y = \infty$.
Case B: $D_y = 16D_x, D_1 = \frac{2}{3}D_x, D_{xy} = \frac{5}{3}D_x, S_x = S_y = \infty$.

Conclusion

Finite strip formulations which involve transverse shear deformation have been presented for the elastic analysis of rectangular and curved plates. The examples presented demonstrate the accuracy and versatility of the formulation.

Nomenclature

a, b	width and length of rectangular plate.
D	flexural rigidity of isotropic plate.
D_x, D_y, D_{xy} D_r, D_t, D_{rt}	plate rigidities in flexure and torsion.
h	plate thickness.
M_x, M_y, M_{xy} M_r, M_t, M_{rt}	bending and twisting moments.
q	distributed transverse loading.
Q_x, Q_y Q_r, Q_t	transverse shearing forces.
r, t	radial and tangential directions, respectively, of curved strip.
s	local natural dimensionless co-ordinate.
S_x, S_y S_r, S_t	transverse shear rigidities of an orthotropic plate.
u, v, w	components of displacement parallel to the x-, y- and z- axes respectively.
x, y, z	rectangular co-ordinates.
α	angle subtended by curved plate.
θ_x, θ_y θ_r, θ_t	normal rotations of plate cross-section.
λ	angular co-ordinate.
ν	Poisson's ratio of isotropic material.
$\{F\}$	nodal force vector.
$\{M\}$	stress resultants vector.
$\{\delta\}$	displacement vector.
$[B]$	matrix connecting strains and displacements of a strip.
$[D]$	property matrix.
$[k]$	stiffness matrix of strip.
$[N]$	$[N_1, N_2, N_3, ...]$ shape function matrix.

Practical Application and Scope

The formulations presented in this paper extend the finite strip method to the analysis of plate structures which undergo considerable transverse shear deformation and cannot therefore be treated by the conventional finite strip approach [1, 2] which is based on customary thin plate theory. Examples of such structures occur frequently in bridge construction. They include sandwich plates and slabs bridges with relatively high depth to span ratio, as well as voided slabs and multicell bridge decks that can be idealized by an equivalent homogeneous material. These structures are being used in increasing numbers in modern highway systems and the application of the finite strip technique to their analysis is of particular interest.

The formulations presented are such that their accuracy is superior to the conventional finite strip and are likely to be adopted as standard in the analysis of straight and curved bridge decks. However, in order to derive their full benefits, care must be taken in programming so as to utilize all possible short cuts and reduce computing time. Reference 4 discusses some useful short cuts that can be achieved in the practical implementation of simply supported finite strips.

Although the present formulation is restricted to simply supported end conditions, Fourier transforms corresponding to a variety of other boundary conditions could be adopted. Also the treatment of intermediate supports follows well established procedures which could be readily incorporated into the formulation.

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Summary

Finite strip formulations are developed for the elastic analysis of transversely loaded rectangular and curved plates with opposite simply supported ends. The formulations involve transverse shear deformation which is included in the analysis by discarding the Kirchoff's normality law used in classical thin plate theory. Numerical examples are presented which demonstrate the applicability of the formulation to thin, thick and sandwich plates.

Résumé

Des formulations par bandes finies sont développées pour l'analyse élastique de plaques rectangulaires et courbes chargées transversalement et supportées aux extrémités. Les formulations comprennent le cisaillement transversal qui est compris dans le calcul en laissant la loi de Kirchhoff de côté telle qu'elle est appliquée dans la théorie classique des plaques minces. Des exemples numériques sont présentés, montrant le champ d'application de la formulation sur des plaques minces, épaisses et sandwich.

Zusammenfassung

Für die elastische Berechnung transversal belasteter rechteckiger und gekrümmter Platten mit entgegengesetzten einfach aufgelagerten Enden werden finite Streifenformulierungen entwickelt. Die Formulierungen schliessen transversale Schubdeformation ein, die in der Berechnung unter Ausserachtlassung des Kirchhoff'schen Normalitätsgesetzes inbegriffen ist, wie dies in der klassischen Theorie dünner Platten verwendet wird. Es werden numerische Beispiele angeführt, welche die Verwendbarkeit der Formulierung für dünne, dicke und Sandwichplatten belegen.