Analysis of curved lattices with generalized joint loadings

Analysis of Curved Lattices with Generalized Joint Loadings

Etude des systèmes réticulés incurvés avec charges nodales généralisées

Untersuchung von gekrümmten Stabsystemen mit verallgemeinerten Knotenlasten

DONALD L. DEAN

Professor of Civil Engineering, University of Delaware, Newark, Delaware

Introduction

In practice, the structural engineer devotes only a small portion of his time to the analysis of single elements such as beams, rings, simple frames, shell forms, etc. More typically he is concerned with a network of members or elements such as a building frame, a truss, a multi-span flat slab system, or a multi-unit shell structure. Under a broad definition of the term, any such network of elements could be termed a lattice, and one might say that the design and analysis of structural lattices is a problem of major concern to the profession. Therefore, a review of the methods used to deal with lattice systems is in order.

The mathematician is surprised to learn that, with rare exception, our literature describes only the crudest tools for dealing with the lattice problem. Numerous relaxation and iteration techniques, as well as automatic computing devices, are used to solve the system of equations, but even these aids become unwieldy when the popular "open form" methods are applied to fine lattices. The branch of mathematics naturally suited for the study of structural lattices is the calculus of finite differences with major emphasis on the sub-topic of difference equations. For a broad and interesting class of structures it is feasible to seek closed form solutions to the governing difference equations. Most structures made up of a number of elements whose geometry and other properties are identical or vary in a regular manner are in this category. Such structures, termed fine or regular lattices, are the object of this study.

Even when a lattice is irregular, or for some other reason must be solved by open methods, the concept of difference equations is useful. It is frequently

possible to "walk through" the solution, that is, start with a specific set of boundary conditions and progress from one unknown to the next until all have been determined. This approach is preferable to the more popular method of solving numerically a system of simultaneous algebraic equations.

The thorough study of fine lattices by BLEICH and MELAN [1] is now more than thirty years old. The subject has received only slight attention by structural engineers in past years and the work of applied mathematicians in the calculus of finite differences has not been directed to the structural field. However, there has been progress in finite mathematics since the publication of this book which is applicable to our field with minor modifications and extensions [2]. The purpose of this paper is twofold: 1 — to revive interest in the use of general closed form solutions for the analysis of structural lattices, and 2 — to present solutions for use in the design and analysis of four typical systems in the fine lattice category.

The Polar Lattice and the Vierendeel Truss

The plane polar lattice symmetrically loaded out of plane and the Vierendeel truss transversely loaded at the joints (see Fig. 1) have identical governing difference equations. This problem is included here for the purpose of introducing new mathematical formulæ free of the geometrical complications encountered with curved structures. It is a limiting case of both the conical lattice and the curved Vierendeel truss, lattices which are analysed in subsequent sections of this paper.
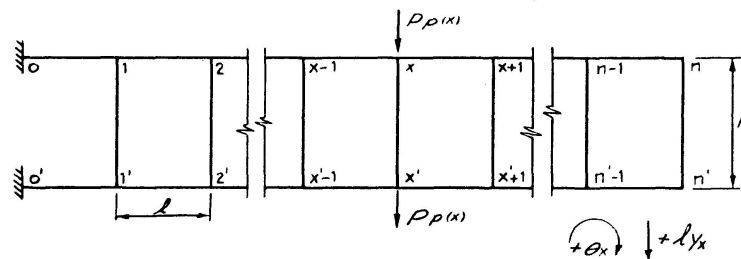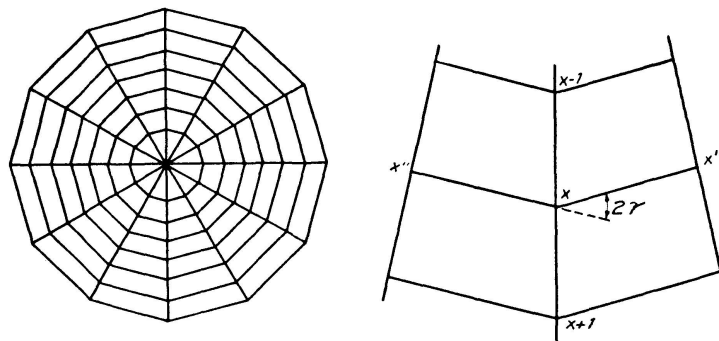

Fig. 1 a. Vierendeel Truss.

Fig. 1 b. Polar Lattice and Typical Element.

Finite Calculus Operators

Only two of the several standard operators will be used here: The Boole displacement operator E, defined by $Ef(x)=f(x+1)$ and $E^n f(x)=f(x+n)$, and the difference operator Δ defined by $\Delta f(x)=f(x+1)-f(x)$ and $\Delta^n f(x)=(E-1)^n f(x)$.

Slope Deflection Equations

The displacement operator is used to express the standard force-deformation relations in forms more suitable for direct solution:

$$M_{x,x+1} = E M_{x-1,x} = 2 K_m [(E+2)\theta_x - 3(E-1)Y_x], \tag{1}$$

$$M_{x+1,x} = E M_{x,x-1} = 2 K_m [(2E+1)\theta_x - 3(E-1)Y_x], \tag{2}$$

$$V_{x,x+1} = E V_{x,x-1} = -\frac{6}{l} K_m [(E+1)\theta_x - 2(E-1)Y_x], \tag{3}$$

where K_m = flexural rigidity of main beam divided by length of typical element, l:

θ_x = joint rotation in radians η,

Y_x = lateral joint movement divided by l.

Equilibrium Equations for Vierendeel Truss

$$M_{x,x+1} + E^{-1} M_{x+1,x} + M_{x,x'} = 0, \tag{4}$$

$$(1-E^{-1}) V_{x,x+1} + P p(x) = 0. \tag{5}$$

Where $M_{x,x'} = 6 K_c \theta_x$, K_c = flexural rigidity of cross member divided by its length h (K_c is here assumed constant), and $P p(x)$ = load at joint x. The governing difference equations are obtained by substituting relations (1), (2), and (3) into (4) and (5) and operating on both sides with E.

$$[E^2 + (4+3\alpha)E + 1]\theta_x - 3[E^2 - 1]Y_x = 0 \tag{6}$$

$$[E^2 - 1]\theta_x - 2[E^2 - 2E + 1]Y_x = C_0 E p(x) \tag{7}$$

where $\alpha = K_c/K_m$, and $C_0 = \dfrac{Pl}{6 K_m}$.

Equilibrium Equations for Polar Lattice

$$M_{x,x+1} + E^{-1} M_{x+1,x} + 2\sin\gamma M_{x,x'} = 0. \tag{8}$$

The Summation of lateral forces at joint x is given by (5): therefore, (7) is one of the governing equations for the polar lattice. $M_{x,x'} = 2 K_c \theta_x \sin\gamma$. Substitution of this relation with (1) and (2) into (8) and operation by E gives

$$[E^2 + (4+3\alpha)E + 1]\theta_x - 3[E^2 - 1]Y_x = 0 \tag{6'}$$

where, here, $\alpha = 2/3 (\sin^2 \gamma) K_c/K_m$.

Load Shape Function

The function $p(x)$ is adimensional and may be chosen to represent any specified variation of joint loading. For example, $p(x) = 1$ represents a uniform or constant loading. When the governing equations are linear, as in this case, the most general possible loading results from the selection of $p(x)$ to denote an impulse loading, i. e. an arbitrarily placed concentrated load. The solution obtained for such a loading can be used to get answers for any other loading by simple superposition, a procedure which is generally easier than reworking the problem for each load situation to be investigated. The Kronecker Delta can be used satisfactorily as an impulse function in finite mathematics. As in the continuous case, the impulse function is closely related to the step function. The following definitions and formulas associated with the Kronecker Delta and the Finite Step Function are used throughout this paper.

Kronecker Delta $\qquad\qquad \delta_x^a = \begin{Bmatrix} 0 & \text{for } x \neq a \\ 1 & \text{for } x = a \end{Bmatrix}.$ $\qquad\qquad$ (9)

Finite Step Function $\qquad U(x-a) = \begin{Bmatrix} 0 & \text{for } x < a \\ 1 & \text{for } x \geq a \end{Bmatrix}.$ $\qquad$ (10)

$$\Delta U(x-a) = \delta_x^{a-1}; \quad = E\,\delta_x^a, \qquad\qquad (11)$$

$$\Delta f(x-a)\,U(x-a) = U(x-a)\,\Delta f(x-a) + f(0)\,\delta_x^{a-1} \qquad (12)$$

$$\Delta^{-1}[U(x-a)\,\Delta f(x-a)] = U(x-a)\,f(x-a) - f(0)\,U(x-a) + C. \qquad (13)$$

Additional formulas for operations on the Finite Step Function are presented in Appendix B.

Solution of Governing Equations

If the loading on either of the lattices considered in this section consists of a load P at joint $x = a$ only, the governing equations are

$$[E^2 + (4 + 3\alpha)\,E + 1]\,\theta_x - 3\,[E^2 - 1]\,Y_x = 0, \qquad\qquad (6')$$

$$[E^2 - 1]\,\theta_x - 2\,[E^2 - 2\,E + 1]\,Y_x = C_0\,\delta_x^{a-1}. \qquad\qquad (7')$$

The operator coefficients of the unknowns θ_x and Y_x are dealt with in the usual algebraic manner to give independent equations

$$(E-1)^2\,(E-\beta)\left(E - \frac{1}{\beta}\right) Y_x = C_0\,[E^2 + (4 + 3\alpha) + 1]\,\delta_x^{a-1}, \qquad (14)$$

$$(E-1)\,(E-\beta)\left(E - \frac{1}{\beta}\right)\theta_x = 3\,(E+1)\,C_0\,\delta_x^{a-1}, \qquad\qquad (15)$$

where $\qquad\qquad\qquad \beta = 1 + 3\alpha + \sqrt{3\alpha(2 + 3\alpha)}.$

It should be noted that the operator determinant of (6) and (7'), the left side of (14), is fourth order so that four boundary conditions will be needed to

evaluate the summation constants in the solution. In following the procedures outlined in [2], seven apparent summation constants appear. The number of independent constants is reduced to four by substituting the results into the dependent system (6) and (7') and matching coefficients of like terms. For convenience in satisfying boundary conditions, hyperbolic functions are preferred for solutions to (14) and (15). Let $e^\varphi = \beta$, or in other terms $1 + 3\alpha = \cosh\varphi$, $\sinh^2\frac{1}{2}\varphi = 1\frac{1}{2}\alpha$ and $\tanh\frac{1}{2}\varphi = \frac{\beta-1}{\beta+1}$. With this transformation, using (2.c) from Appendix B for the particular solutions and routine procedures for the homogeneous solutions, the total solution for the system is

$$\frac{2\alpha}{C_0}Y_x = C_1 + C_2 x + C_3\cosh\varphi x + C_4\sinh\varphi x$$
$$+ \left[\coth\frac{\varphi}{2}\sinh\varphi(x-a) - (2+\alpha)(x-a)\right]U(x-a), \tag{16}$$

$$\frac{2\alpha}{C_0}\theta_x = \frac{2}{2+\alpha}C_2 + 2\tanh\frac{\varphi}{2}[C_4\cosh\varphi x + C_3\sinh\varphi x]$$
$$+ 2[\cosh\varphi(x-a) - 1]U(x-a). \tag{17}$$

Boundary Conditions

Eqs. (16) and (17) contain complete and general solutions for systems having (6) and (7') as mathematical models. The four summation constants C_1, C_2, C_3 and C_4 can be evaluated by substituting the appropriate boundary conditions into the solutions. This step will be illustrated by particularizing (16) and (17) to the mixed boundary conditions of the cantilever. The cantilevered Vierendeel truss is shown in Fig. 1. For a symmetrical load, cantilever boundary conditions refer to the "umbrella" polar grid, a single support at the center with no restraints at the perimeter. The cantilevered lattice must satisfy the following four boundary conditions:

$$\overset{1}{\theta_0} = \overset{2}{Y_0} = \overset{3}{M_{n,n+1}} = \overset{4}{V_{n,n+1}} = 0.$$

Boundary conditions three and four may be used more easily if the general solution is augmented by two more equations. Substituting (16) and (17) into (1) and (3), the expressions for joint moments and shears are found to be:

$$\frac{-2}{Pl}M_{x,x+1} = \frac{1}{2+\alpha}C_2 + \mathrm{sech}\frac{\varphi}{2}[C_3\cosh\varphi(x+\tfrac{1}{2}) + C_4\sinh\varphi(x+\tfrac{1}{2})]$$
$$+ \left[\mathrm{csch}\frac{\varphi}{2}\sinh\varphi(x-a+\tfrac{1}{2}) - 1\right]U(x-a), \tag{18}$$

$$V_{x,x+1} = \frac{P}{2+\alpha}C_2 - PU(x-a). \tag{19}$$

Inserting the four cantilever boundary conditions into (17), (16), (18) and (19) respectively,

1. $\dfrac{1}{2+\alpha} C_2 + \tanh \dfrac{\varphi}{2} C_4 = 0.$

2. $C_1 + C_3 = 0.$

3. $\dfrac{1}{2+\alpha} C_2 + C_{\varphi,n} C_3 + D_{\varphi,n} C_4 = 1 - \coth \dfrac{\varphi}{2} D_{\varphi,n-a}.$

4. $C_2 = 2+\alpha,$

where

$$C_{\varphi,n} = \operatorname{sech} \dfrac{\varphi}{2} \cosh \varphi \, (n + \tfrac{1}{2}) = \tanh \dfrac{\varphi}{2} \sinh \varphi \, n + \cosh \varphi \, n,$$

$$D_{\varphi,n} = \operatorname{sech} \dfrac{\varphi}{2} \sinh \varphi \, (n + \tfrac{1}{2}) = \tanh \dfrac{\varphi}{2} \cosh \varphi \, n + \sinh \varphi \, n$$

and $C_{\varphi,n-a}, D_{\varphi,n-a}$ are identical except n is replaced by $n-a$.

Now that the summation constants have been evaluated for a specific set of boundary conditions, the final working formulae for the various lattice variables may be written.

$$\dfrac{2\alpha}{C_0} Y_x = \coth \dfrac{\varphi}{2} F_{\varphi n a} (\cosh \varphi x - 1) - \coth \dfrac{\varphi}{2} [\sinh \varphi x - U(x-a) \sinh \varphi (x-a)]$$
$$(2+\alpha)[x - (x-a) U(x-a)], \tag{16a}$$

$$\dfrac{\alpha}{C_0} \theta_x = F_{\varphi n a} \sinh \varphi x - (\cosh \varphi x - 1) + [\cosh \varphi (x-a) - 1] U(x-a), \tag{17a}$$

$$\dfrac{-2}{Pl} M_{x,x+1} = 1 - U(x-a) + \operatorname{csch} \dfrac{\varphi}{2} [F_{\varphi n a} \cosh (x + \tfrac{1}{2}) - \sinh \varphi (x + \tfrac{1}{2})$$
$$+ U(x-a) \sinh \varphi (x-a+\tfrac{1}{2})], \tag{18a}$$

$$V_{x,x+1} = P[1 - U(x-a)]. \tag{19a}$$

Where

$$F_{\varphi n a} = \dfrac{1}{C_{\varphi,n}} (D_{\varphi,n} - D_{\varphi,n-a}) = \operatorname{sech} \varphi \, (n + \tfrac{1}{2})[\sinh \varphi \, (n + \tfrac{1}{2}) - \sinh \varphi \, (n + \tfrac{1}{2} - a)].$$

Expressions (16a) thru (19a) comprise a complete solution to the fundamental problem associated with the cantilevered polar grid or Vierendeel truss, namely the case of the arbitrarily placed single joint load. Results for any set of joint loads can be obtained from this solution by superposition of results for different values of a. The case of loads placed between joints can be dealt with by first solving the simple case in which the joints are considered to be restrained against translation and adding the effects of joint loads which cancel the restraining forces. In some practical applications, a pattern of joint loads will be repeated frequently enough so that the stress analyst will prefer to operate on the general solution to obtain expressions applicable to a specific

load pattern, rather than make repeated substitutions into the general solution. One example of a load pattern which occurs frequently enough to warrant a special solution is the case of $p(x) = U(x - a')$, a uniform series of joint loads beginning at $x = a'$. This loading includes the case of uniform joint loads over the entire structure as a special case where $a' = 1$.

To illustrate the manner in which one operates on the general solution to obtain expressions needed for different load patterns, the case of the "step" loading will be carried to the point of working formulas and illustrated numerically below. The desired formulas are obtained by summing the right hand expressions in (16 a), (17 a), (18 a) and (19 a) (functions of the parameter a as well as the variable x) over all values of a from a' to n. The new quantities will be denoted with a prime (') to distinguish them from the corresponding quantities for the general loading.

$$Y'_{x,a'} = \sum_{a=a'}^{a=n} Y_{x,a} = \Delta_a^{-1} Y_{x,a}\big|_{a=a'}^{a=n+1}, \tag{16b}$$

$$\theta'_{x,a'} = \sum_{a=a'}^{a=n} \theta_{x,a} = \Delta_a^{-1} \theta_{x,a}\big|_{a=a'}^{a=n+1}, \tag{17b}$$

$$M'_{x,a'} = \sum_{a=a'}^{a=n} M_{x,x+1} = \Delta_a^{-1} M_{x,x+1}\big|_{a=a'}^{a=n+1}, \tag{18b}$$

$$V'_{x,a'} = \sum_{a=a'}^{a=n} V_{x,x+1} = \Delta_a^{-1} V_{x,x+1}\big|_{a=a'}^{a=n+1}. \tag{19b}$$

Δ_a^{-1}, signifies the inverse difference, or summation operation, with respect to the parameter a. In other words if $\Delta_a^{-1} f(x-a) = F(x-a)$, then $F(x-a-1) - F(x-a) = f(x-a)$. One must exercise care in using formulas (11), (12) and (13) if the operations are with respect to a rather than x. The following modified forms were used in carrying out the operations indicated in (16 b) thru (19 b):

$$\Delta_a U(x-a) = -\delta_x^a; \quad \Delta_a f(x-a) U(x-a) = U(x-a) \Delta_a f(x-a) - f(-1) \delta_x^a, \tag{11a}$$

$$\Delta_a^{-1} [U(x-a) \Delta_a f(x-a)] = [f(x-a) - f(-1)] U(x-a) + C. \tag{12a}$$

Working formulas for the various quantities needed in the analysis of the polar grid or Vierendeel truss subjected to a "step" joint loading are obtained by using (12 a) along with standard inverse difference formulas as indicated in (16 b) thru (19 b). After some simplification, the results are

$$\frac{2\alpha}{C_0} Y'_x = F'_{\varphi n a'} \frac{\cosh \varphi x - 1}{\tanh \frac{1}{2}\varphi} + (n+1-a') \left[(2+\alpha)x - \coth \frac{\varphi}{2} \sinh \varphi x \right]$$
$$+ \frac{1}{2} \left[\frac{2+3\alpha}{3\alpha} (C_{\varphi, x-a'} - 1) - (2+\alpha)(x+1-a')_2 \right] U(x-a'), \tag{16c}$$

$$\frac{\alpha}{C_0} \theta'_x = F'_{\varphi na'} \sinh \varphi x - (n + 1 - a')(\cosh \varphi x - 1)$$
$$+ \left[\frac{1}{2} \coth \frac{\varphi}{2} D_{\varphi, x-a'} - (x + \tfrac{1}{2} - a')\right] U(x - a'), \tag{17c}$$

$$\frac{2}{Pl} M'_{x,x+1} = (n + 1 - a') \left[\frac{\sinh \varphi (x + \tfrac{1}{2})}{\sinh \tfrac{1}{2} \varphi} - 1\right] - (F'_{\varphi na'}) \frac{\cosh \varphi (x + \tfrac{1}{2})}{\sinh \tfrac{1}{2} \varphi}$$
$$+ (x + 1 - a') U(x - a') - \frac{1}{3\alpha} [\cosh \varphi (x + 1 - a') - 1] U(x - a'), \tag{18c}$$

$$V'_{x,x+1} = P[(n + 1 - a') - (x + 1 - a') U(x - a')], \tag{19c}$$

where $$F'_{\varphi na'} = \frac{1}{C_{\varphi,n}} \left[(n + 1 - a') D_{\varphi,n} - \frac{\cosh \varphi (n + 1 - a') - 1}{\sinh \varphi}\right].$$

To facilitate an understanding of the use of the solutions presented in this section, formulas (16 c) thru (19 c) will be illustrated with a numerical example. Take the case of a timber Vierendeel truss of six panels, each 6.0 feet by 4.0 feet with 3×10 main members and 2×8 cross members, loaded with 1.0 kip joint loads beginning at joint two.

$l = 6.0;$ $h = 4.0;$ $I_m = 187.55;$ $I_c = 57.13;$ $a' = 2;$ $P = 0.5;$

$n = 6;$ $\alpha = 0.456918;$ $\beta = 2.37075 + 2.14953 = 4.52028;$

$C_0 = 0.001309;$ $\varphi = 1.50858;$ $\tanh \frac{\varphi}{2} = \frac{1.37075}{2.14953} = 0.637700;$

$C_{\varphi n} = 6985.593 = D_{\varphi n};$ $F'_{\varphi na'} = 4.937224.$

$$\frac{2\alpha}{C_0} Y'_x = 7.74224 (\cosh \varphi x - 1) - 7.84068 (\sinh \varphi x - 1.56678 x)$$
$$+ 1.22953 [C_{\varphi, x-2} - 1 - 0.999133 (x - 1)_2] U(x - 2), \tag{16d}$$

$Y'_6 = 0.05817$ or end deflection $= 4.19$ inches.

$$\frac{\alpha}{C_0} \theta'_x = F'_{\varphi na'} \sinh \varphi x - 5 (\cosh \varphi x - 1) + [0.784068 D_{\varphi, x-2} - x + \tfrac{3}{2}] U(x - 2), \tag{17d}$$

$\theta'_6 = 0.002242$ or $0.128°.$

$$\frac{2}{Pl} M'_{x,x+1} = 5 \left[\frac{\sinh \varphi (x + \tfrac{1}{2})}{0.827876} - 1\right] - 5.96372 \cosh \varphi (x + \tfrac{1}{2}) + (x - 1) U(x - 2)$$
$$- \left[\frac{\cosh \varphi (x - 1) - 1}{1.370754}\right] U(x - 2), \tag{18d}$$

$M_{56} = 0.6497 k';$ $M_{01} = 14.61 k'.$

For certain ranges of the governing parameters and variables, the results given in this section are sensitive and require more than the usual number of significant figures in the intermediate calculations for satisfactory engineering answers. If frequent use of the formulas in these ranges is required, it would probably be more convenient to transform them to an asymptotic form.

The Symmetrically Loaded Conical Lattice

The cone is a surface used frequently by structural designers to shelter circular areas and is well suited for a lattice construction method. Although nearly an endless number of lattice systems could be used to approximate or
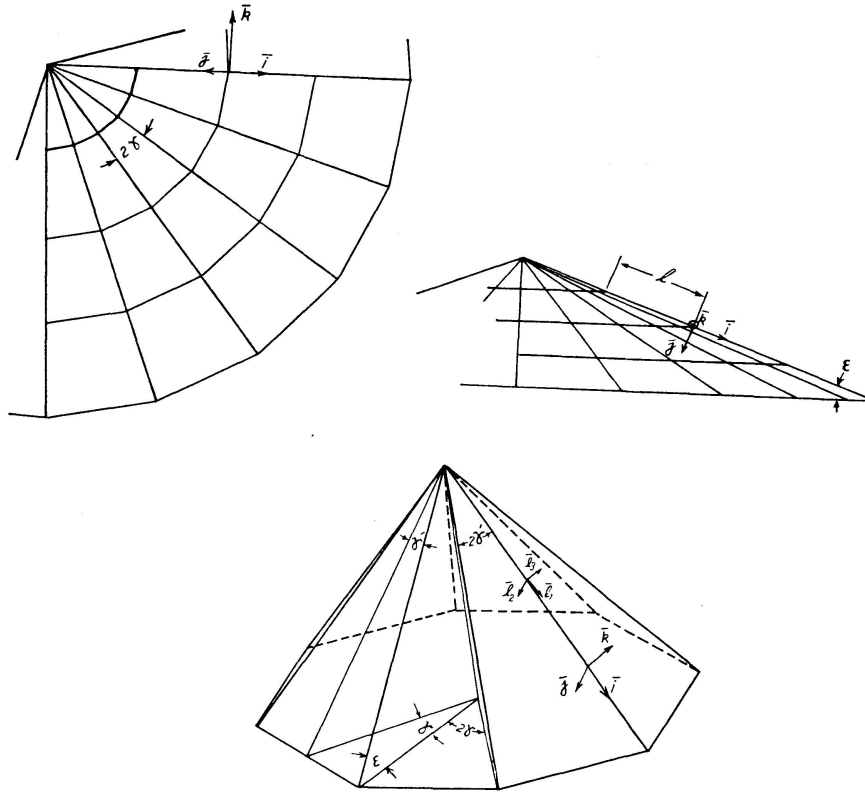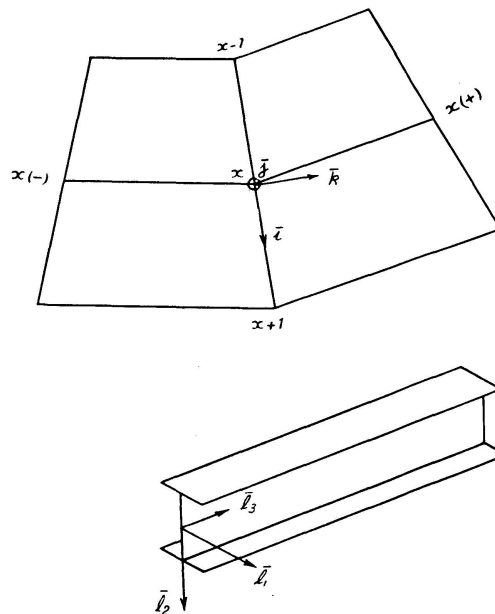

Fig. 2. Conical Lattice.

Fig. 3. Typical Element and Member Reference Frame for Conical Lattice.

coincide with a conical surface, the system consisting of main members radiating from the apex and cross members lying in a plane perpendicular the axis (see Fig. 2 and 3) is the one which first comes to mind. This network would seem to be fundamental for the shape under consideration as its members follow closely the lines of principal curvature for the continuous surface.

In analysing this lattice, the usual assumptions made for a frame problem will be relaxed regarding direct stress in the cross members. A small change in length of the cross members has a magnified effect on joint deformations for relatively flat surfaces which will be accounted for in the force-deformation relations. The effects of direct stress on curvature in individual cross members is ignored.

The conical lattice is a space network requiring a three dimensional form of the force-deformation relations (1) and (2). This is conveniently written by using the standard notations of vector algebra.

$$M_{bc} = 2\,K_{bc}([2\,\bar{\theta}_b + \bar{\theta}_c]\cdot\bar{l}_1 + 3\,(\overline{Y}_c - \overline{Y}_b)\cdot\bar{l}_2] + M_{Fbc}, \tag{20}$$

$$M'_{bc} = 2\,K'_{bc}[(2\,\bar{\theta}_b + \bar{\theta}_c)\cdot\bar{l}_2 - 3\,(\overline{Y}_c - \overline{Y}_b)\cdot\bar{l}_1] + M'_{Fbc}, \tag{21}$$

$$T_{bc} = J_{bc}\,(\bar{\theta}_b - \bar{\theta}_c)\cdot\bar{l}_3 + T_{Fbc}, \tag{22}$$

$$S_{bc} = Q_{bc}\,(\overline{Y}_c - \overline{Y}_b)\cdot\bar{l}_3. \tag{23}$$

M_{bc}, M'_{bc}, and T_{bc} are joint moment components parallel respectively to the beam reference axes $\bar{l}_1$, $\bar{l}_2$, and $\bar{l}_3$. They are assumed positive when acting on the beam in accordance with the right hand vector rule. M_{Fbc}, M'_{Fbc} and T_{Fbc} are the corresponding moments caused by mid-span loads if joint deformations are prevented. K_{bc} and K'_{bc} are respectively the flexural rigidities about $\bar{l}_1$ and $\bar{l}_2$ divided by length of member. J_{bc} equals torsional rigidity divided by length of member. Q_{bc} equals axial rigidity or product of cross-sectional area and modulus of elasticity. $\bar{\theta}$ and $\overline{Y}$ are adimensional joint deformations in vector form; $\bar{\theta}$ = three components of rotation and $\overline{Y}$ = three components of translation divided by length of member. $\bar{l}_1$, $\bar{l}_2$ and $\bar{l}_3$ are the three unit reference vectors for a given member coinciding respectively with the major and minor bending axis and the longitudinal axis. It is usually convenient to relate the member reference vectors to a set of base vectors as follows:

$$\bar{l}_1 = l_{11}\bar{i} + l_{12}\bar{j} + l_{13}\bar{k}; \qquad \bar{l}_2 = l_{21}\bar{i} + l_{22}\bar{j} + l_{23}\bar{k}; \qquad \bar{l}_3 = l_{31}\bar{i} + l_{32}\bar{j} + l_{33}\bar{k}. \tag{24}$$

The center dot ($\cdot$) denotes scalar vector multiplication.

The specific problem considered here is the conical lattice with members oriented so that $\bar{l}_2$ for the main members is normal to the circumscribing cone and $\bar{l}_2$ for the cross members is normal to plane surfaces outlined by the lattice elements or normal to the inscribed cone midway between the main lattice members.

Lattice Geometry

$$\sin \gamma' = \sin \gamma \cos \epsilon . \tag{25}$$

Main or radial Member $\quad \bar{l}_1 = -\bar{k}; \quad \bar{l}_2 = \bar{j}; \quad \bar{l}_3 = \bar{i}.$ (26)

Cross Member $\quad \bar{l}_{1c} = \cos \gamma' \, \bar{i} + \tan \gamma' \sin \gamma \sin \epsilon \, \bar{j} + \tan \gamma' \cos \gamma \, \bar{k};$

$$\bar{l}_{2c} = 0 + \frac{\cos \gamma}{\cos \gamma'} \bar{j} - \tan \gamma' \tan \epsilon \, \bar{k}; \tag{27}$$

$$\bar{l}_{3c} = -\sin \gamma' \, \bar{i} + \sin \gamma \sin \epsilon \, \bar{j} + \cos \gamma \, \bar{k}.$$

Deformation Vectors — (from axial symmetry)

$$\bar{\theta}_x \quad = 0 + 0 + \theta_x \bar{k}; \qquad \bar{Y}_x = 0 + l\,Y_x \bar{j} + 0 ;$$

$$\bar{\theta}_{(+x)} = \theta_x \left[-\sin 2\gamma \cos \epsilon \, \bar{i} + \sin 2\gamma \sin \epsilon \, \bar{j} + \cos 2\gamma \, \bar{k} \right]; \tag{28}$$

$$\bar{Y}_{(+x)} = l\,Y_x \left[\sin^2 \gamma \sin 2\epsilon \, \bar{i} + (1 - 2\sin^2 \gamma \sin^2 \epsilon) \bar{j} - \sin 2\gamma \sin \epsilon \, \bar{k} \right].$$

Force-Deformation Relations

Substitution of expressions (27) and (28) into the general space slope deflection Eqs. (20) thru (23), carrying out the indicated operations and simplification gives the following force-deformation relations for axisymmetric loading:

$$M_{x,x+1} = E\,M_{x-1,x} = 2\,K_m \left[-(E+2)\,\theta_x + 3\,(E-1)\,Y_x \right],$$
$$M_{x+1,x} = E\,M_{x,x-1} = 2\,K_m \left[-(2\,E+1)\,\theta_x + 3\,(E-1)\,Y_x \right], \tag{29}$$

$$M_{x,(+x)} = -M_{(+x),x} = 2\,K_c (\tan \gamma' \cos \gamma)\,\theta_x , \tag{30}$$

$$M'_{x,(+x)} = -M'_{(+x),x} = -2\,K'_c (\tan \gamma' \tan \epsilon)\,\theta_x , \tag{31}$$

$$S_{x,(+x)} \quad = -2\frac{l}{h}\,Q_c (\sin \gamma \sin \epsilon)\,Y_x , \tag{32}$$

$$M'_{x,x+1} = T_{x,x+1} = T_{x,(+x)} = T_{(+x),x} = 0 \quad \text{(from axisymmetry)} .$$

Equilibrium Equations

The two necessary equilibrium equations are obtained by summing $\bar{k}$ moments and $\bar{j}$ forces acting on joint x.

$$-M_{x,x+1} - E^{-1} M_{x+1,x} + 2\,M_{x,(+x)} l_{13} + 2\,M'_{x,(+x)} l_{23} = 0 , \tag{33}$$

$$(1 - E^{-1})\,M_{x,x+1} + (1 - E^{-1})\,M_{x+1,x} + 2\,l\,S_{x,(+x)} l_{32} + l\,P\,p\,(x) = 0 , \tag{34}$$

where l is length of the main member between cross beam connections, P is the reference joint load and $p(x)$ is the load shape function. Substitution of relations (29) thru (32) into (33) and (34), after operation with E and simplification, yield the governing difference equations for the symmetrically loaded conical lattice shown in Figs. [2] and [3].

$$[E^2 + 4(1+\alpha)E + 1]\theta_x - 3[E^2 - 1]Y_x = 0, \tag{33a}$$

$$[E^2 - 1]\theta_x - 2[E^2 - 2(1+\zeta)E + 1]Y_x = C_0 E p(x), \tag{34a}$$

where $\quad \alpha = \frac{1}{2}\tan^2\gamma' \frac{K_c}{K_m}\left(\cos^2\gamma + \frac{K'_c}{K_c}\tan^2\epsilon\right), \qquad C_0 = \frac{l\,P}{6\,K_m}$

and $\quad \zeta = \frac{l^2 Q_c}{6\,h\,K_m}\sin^2\gamma \sin^2\epsilon.$

As in problem one, the loading is assumed to be an arbitrarily located series of joint loads on a circle at $x = a$. In other words, the load function $p(x) = \delta_x^a$ — the finite impulse function. Then, solving (33 a) and (34 a) for θ_x and Y_n, one obtains the following independent form of the equations.

$$(E^2 - 2jE + 1)(E^2 - 2kE + 1)\theta_x = 3C_0 E(E^2 - 1)\delta_x^a, \tag{35}$$

$$(E^2 - 2jE + 1)(E^2 - 2kE + 1)Y_x = C_0 E[E^2 + 4(1+\alpha)E + 1]\delta_x^a. \tag{36}$$

For practical ranges of the parameters α and ζ, j and k are a conjugate pair of complex numbers, given by

$$j, k = (1 + 2\alpha - \zeta) \pm i\sqrt{\zeta(6-\zeta) + 4\alpha(2\zeta - \alpha)}$$

or $\quad \sqrt{4(1+\alpha)(1+\zeta) - 3}\, exp \pm i\cos^{-1}\dfrac{1 + 2\alpha - \zeta}{\sqrt{4(1+\alpha)(1+\zeta) - 3}}.$

As shown in Appendix A, the solutions for (35) and (36) can be written:

$$\frac{1}{C_0}\theta_x = C_1\cosh\eta x\cos\varphi x + C_2\cosh\eta x\sin\varphi x$$
$$+ C_3\sinh\eta x\cos\varphi x + C_4\sinh\eta x\sin\varphi x + \theta_{xp}, \tag{35a}$$

$$\frac{1}{C_0}Y_x = C'_1\cosh\eta x\cos\varphi x + C'_2\cosh\eta x\sin\varphi x$$
$$+ C'_3\sinh\eta x\cos\varphi x + C'_4\sinh\eta x\sin\varphi x + Y_{xp}, \tag{36a}$$

where e^η and φ are the amplitude and argument respectively for the complex number $j + \sqrt{j^2 - 1}$ in polar form. $C_1 \ldots C'_4$ are the summation constants evaluated by use of the boundary conditions. These constants comprise only four independent quantities as may be determined by substituting the solutions (35 a) and (36 a) into either of the dependent Eqs. (33 a) and (34 a). This operation yields the following relations:

$$C'_1 = C_{\alpha\zeta}C_3 - D_{\alpha\zeta}C_2, \qquad C'_2 = D_{\alpha\zeta}C_1 + C_{\alpha\zeta}C_4,$$
$$C'_3 = C_{\alpha\zeta}C_1 - D_{\alpha\zeta}C_4, \qquad C'_4 = D_{\alpha\zeta}C_3 + C_{\alpha\zeta}C_2,$$

$$C_{\alpha\zeta} = \frac{2}{3}\frac{\sinh\eta\,[\cosh\eta + 2(1+\alpha)\cos\varphi]}{\cosh 2\eta - \cos 2\varphi}$$

or $\quad \dfrac{1}{2}\dfrac{\sinh\eta\,[\cosh\eta - (1+\zeta)\cos\varphi]}{[\cosh\eta - (1+\zeta)\cos\varphi]^2 + \zeta(2+\zeta)\sin^2\varphi},$

$$D_{\alpha\zeta} = \frac{2}{3} \frac{\sin\varphi \, [2\,(1+\alpha)\cosh\eta + \cos\varphi]}{\cosh 2\eta - \cos 2\varphi}$$

$$\text{or} \quad \frac{1}{2} \frac{\sin\varphi \, [(1+\zeta)\cosh\eta - \cos\varphi]}{[\cosh\eta - (1+\zeta)\cos\varphi]^2 + \zeta\,(2+\zeta)\sin^2\varphi} \, .$$

θ_{xp} and Y_{xp} are the particular sums, denoted operationally

$$\theta_{xp} = \frac{3\,E\,(E^2-1)}{(E^2-2jE+1)(E^2-2kE+1)} \delta_x^a$$

$$= \frac{3\,\varDelta^2\,(E+1)}{(E^2-2jE+1)(E^2-2kE+1)} U\,(x-a) \tag{35 b}$$

$$\text{and} \quad Y_{xp} = \frac{\varDelta\,[E^2+4\,(1+\alpha)\,E+1]}{(E^2-2jE+1)(E^2-2kE+1)} U\,(x-a). \tag{36 b}$$

Formulas for "division" by the operator appearing in the denominator are given in Appendix B. Substitution into the appropriate formula and operation as indicated in the numerator yield the particular sums needed to complete the general solutions (35 a) and (36 a).

$$\theta_{xp} = \frac{3\,U\,(x-a)}{\sinh\eta \, \sin\varphi} \sinh\eta\,(x-a)\sin\varphi\,(x-a), \tag{35 c}$$

$$Y_{xp} = \frac{3\,U\,(x-a)}{\sinh\eta \, \sin\varphi} [C_{\alpha\zeta}\cosh\eta\,(x-a)\sin\varphi(x-a)$$
$$- D_{\alpha\zeta}\sinh\eta\,(x-a)\cos\varphi\,(x-a)]. \tag{36 c}$$

The last two expressions complete the general closed form solutions for the deformations of the joints on a symmetrically loaded conical lattice. Any symmetrical system of joint loads can be dealt with by simple superposition of the above results for a "ring" of joint loads at $x=a$. In some cases, the superposition can be carried out conveniently by operating on the general solution as illustrated in problem one. The force components needed for a stress analysis are obtained by substituting expressions for the deformations into formulas (29) thru (32). The formula for joint moments in the main members can be obtained by substitution of the above solutions into (29), giving

$$\frac{3}{Pl} M_{x,x+1} = \frac{2\alpha}{C_0}\theta_x + \frac{1+2\alpha}{\cosh\eta + \cos\varphi} \Big[(C_3\sinh\eta + C_2\sin\varphi)\cosh\eta\,x\cos\varphi\,x$$

$$+ (C_2\sinh\eta - C_3\sin\varphi)\sinh\eta\,x\sin\varphi\,x + (C_4\sinh\eta\,h - C_1\sin\varphi\,x)\cosh\eta\,x\sin\varphi\,x$$

$$+ (C_1\sinh\eta + C_4\sin\varphi)\sinh\eta\,x\cos\varphi\,x + 3\,\frac{\sinh\eta\,(x-a)\cos\varphi\,(x-a)}{\sinh\eta} U\,(x-a)$$

$$+ 3\,\frac{\cosh\eta\,(x-a)\sin\varphi\,(x-a)}{\sin\varphi} U\,(x-a) \Big]. \tag{37}$$

Boundary Conditions

As in problem one, the four summation constants $C_1 \ldots C_4$ can be determined for a particular case by substituting the appropriate boundary conditions into the general solutions. The plausible boundary conditions, $\theta_0 = Y_0 = = Y_n = M_{n,n+1} = 0$, are used to give: $C_1 = 0$, $D_{\alpha\zeta} C_2 = C_{\alpha\zeta} C_3$,

$$C_3 = 3 D_{\alpha\zeta} \frac{F_n \sinh \eta \, a \sin \varphi \, (n-a) - G_n \sinh \eta \, (n-a) \sin \varphi \, a}{\sinh \eta \sin \varphi \, (C_{\alpha\zeta} F_n \sin \varphi \, n - D_{\alpha\zeta} G_n \sinh \eta \, n)} \, ,$$

$$C_4 = 3 C_{\alpha\zeta} \frac{G_n \cosh \eta \, (n-a) \sin \varphi \, a - F_{n-a} \cosh \eta \, a \sin \varphi \, n}{\sinh \eta \sin \varphi \, (C_{\alpha\zeta} F_n \sin \varphi \, n - D_{\alpha\zeta} G_n \sinh \eta \, n)} \, ,$$

$$- 3 D_{\alpha\zeta} \frac{F_n \sinh \eta \, a \cos \varphi \, (n-a) - G_{n-a} \sinh \eta \, n \cos \varphi \, a}{\sinh \eta \sin \eta \, (C_{\alpha\zeta} F_n \sin \varphi \, n - D_{\alpha\zeta} G_n \sinh \eta \, n)} \, ,$$

where $F_n = 3 \alpha C_{\alpha\zeta} (\cosh 2\eta - \cos 2\varphi) \sin \varphi \, n + (1 + 2\alpha)(3 + 2\alpha) \sinh \eta \sin \varphi \cos \varphi \, n$
$G_n = 3 \alpha D_{\alpha\zeta} (\cosh 2\eta - \cos 2\varphi) \sinh \eta \, n + (1 + 2\alpha)(3 + 2\alpha) \sinh \eta \sin \varphi \cosh \eta \, n$ and F_{n-a} and G_{n-a} are the corresponding expressions with n replaced by $n-a$.

Numerical Example

To check the general solutions (35 a) thru (37), and the summation constants above, the closed form expressions were used to obtain numerical results for a relatively coarse lattice. These answers were then compared with the corresponding results obtained by use of an open form approach to validate the work in this section. The results of these numerical calculations are listed here for the purpose of demonstrating the use of closed form expressions in the analysis of conical lattices.

Assume 18 main members of 3 by 14 inch timbers, 40 foot in length, and one set of cross members of 2 by 4 inch stock. Thus: $\gamma = 10^\circ$, $\epsilon = 30^\circ$, $l = 20'$, $n = 2$, and $a = 1$; then $\alpha = 0.00048$ and $\zeta = 2.942$, or $j, k = -1.94152 \pm i \, 3.00132$ and $j + \sqrt{j^2 - 1} = -3.80974 + i \, 6.12040$ so that $e^\eta = 7.20926$ and $\cos \varphi = -0.528451$, or $\sinh \eta = 3.53528$, $\cosh \eta = 3.67399$ and $\sin \varphi = 0.848964$. These basic quantities are combined as indicated above to give: $C_{\alpha\zeta} = 0.233260$, $D_{\alpha\zeta} = 0.146067$, $F_n = F_2 = -3.98813$, $F_{n-a} = F_1 = -4.75670$, $G_2 = 234.514$, $G_1 = 33.1425$, $C_3 = 0.11756$, $C_2 = 0.18774$, $C_4 = -0.11178$, $Y_1 = 0.07114 C_0$, $\theta_1 = 0.03044 C_0$, $\theta_2 = -0.1218 C_0$, $M_{0,1} = -0.06099 P l$, and $M_{1,2} = 0.05084 P l$.

The Arched Truss

The regular arched triangulated truss whose joints lie along the arcs of two concentric circles is a lattice which has been used frequently by structural engineers, but presents a stress and deflection problem which has not been

solved in a closed form suitable for use in analyzing the system with a variety of parameters representing the governing physical constants. The work in this section is directed toward furnishing such a solution. This problem comprises the first step toward the analysis of a cylindrical space lattice with circularly arched trusses as transverse ribs.

Equilibrium Equations

Summation of radial and tangential forces on the inner and outer chord joints of the lattice element (see Fig. 4) furnish the four basic equations needed to analyse this system.
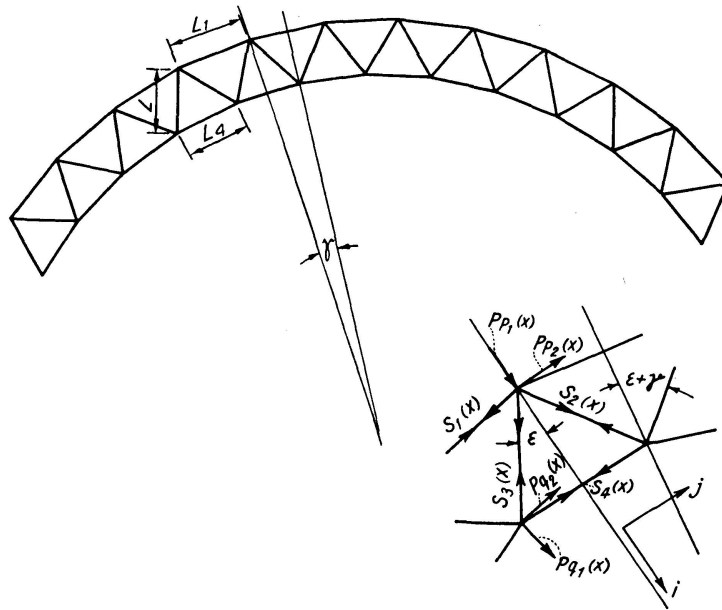
Fig. 4. Arched Truss and Typical Element.

$$\sin \gamma \, (1 + E^{-1}) \, S_1 \, (x) + \cos \epsilon \, [S_2 \, (x) + S_3 \, (x)] + P \, p_1 \, (x) = 0, \tag{41}$$

$$\cos \gamma \, (1 - E^{-1}) \, S_1 \, (x) + \sin \epsilon \, [S_2 \, (x) - S_3 \, (x)] + P \, p_2 \, (x) = 0, \tag{42}$$

$$\sin \gamma \, (1 + E^{-1}) \, S_4 \, (x) - \cos (\epsilon + \gamma) \, [S_3 \, (x) + E^{-1} \, S_2 \, (x)] + P \, q_1 \, (x) = 0, \tag{43}$$

$$\cos \gamma \, (1 - E^{-1}) \, S_4 \, (x) + \sin (\epsilon + \gamma) \, [S_3 \, (x) - E^{-1} \, S_2 \, (x)] + P \, q_2 \, (x) = 0. \tag{44}$$

Force Deformation Relations

As with the first two problems, the approach used will be one of unknown joint deformations. Therefore, relations are needed between the unknown bar forces appearing in equations (41) thru (44) and the joint deformations desired as unknowns. Let $\overline{U} \, (x)$ denote the vector deformation of the outer joint x and $\overline{V} \, (x)$ denote the same quantity for the corresponding inner joint. Then the four unknown scalar quantities are the radial and tangential components of these two plane vector deformations, $U_1 \, (x)$, $U_2 \, (x)$, $V_1 \, (x)$ and $V_2 \, (x)$. If $\bar{i}$

and $\bar{j}$ are, respectively, the unit radial and tangential vectors at the outer joint x, then the unknown deformation components can be identified as follows:

$$\bar{U}(x) = U_1(x)\,\bar{i} + U_2(x)\,\bar{j}$$

and $\qquad \bar{V}(x) = [\cos\gamma\,V_1(x) - \sin\gamma\,V_2(x)]\,\bar{i} + [\sin\gamma\,V_1(x) + \cos\gamma\,V_2(x)]\,\bar{j}.$

The basic form of Hooke's Law, as studied in elementary courses on strength of materials, is employed to write the needed force-deformation relations as follows:

$$S_1(x) = \frac{Q_1}{L_1}\,[(E-1)\,\bar{U}(x)]\cdot\bar{l}_1, \qquad\qquad S_2(x) = \frac{Q}{L}[E\,\bar{V}(x) - \bar{U}(x)]\cdot\bar{l}_2,$$

$$S_3(x) = \frac{Q}{L}[\bar{V}(x) - \bar{U}(x)]\cdot\bar{l}_3 \quad \text{and} \quad S_4(x) = \frac{Q_4}{L_4}\,[(E-1)\,\bar{V}(x)]\cdot\bar{l}_4,$$

where Q, Q_1, and Q_4, respectively, denote the axial rigidities (products of areas and moduli of elasticity) of the diagonals, outer chord members and inner chord members, L, L_1 and L_4 are the lengths of the corresponding members, $\bar{l}_1$, $\bar{l}_2$, $\bar{l}_3$ and $\bar{l}_4$ are unit vectors parallel to the truss members having the same subscript, i. e., $\bar{l}_1 = \sin\gamma\,\bar{i} + \cos\gamma\,\bar{j}$, $\bar{l}_2 = \cos\epsilon\,\bar{i} + \sin\epsilon\,\bar{j}$, $\bar{l}_3 = \cos\epsilon\,\bar{i} - \sin\epsilon\,\bar{j}$ and $\bar{l}_4 = \bar{j}$.

Governing System of Equations

Substituting the relations above into (41) thru (44) and performing the indicated operations, one obtains the following system of equations in terms of unknown joint deformation components:

$$-\left[\frac{\alpha\sin^2\gamma}{2\sin(\epsilon+\gamma)}(E+1)^2 + 2\cos^2\epsilon\,E\right]U_1(x) + \frac{\alpha\sin 2\gamma}{4\sin(\epsilon+\gamma)}(E^2-1)\,U_2(x) \qquad (41\,\text{a})$$

$$+\cos\epsilon\cos(\epsilon+\gamma)\,E\,(E+1)\,V_1(x) + \cos\epsilon\sin(\epsilon+\gamma)\,E\,(E-1)\,V_2(x) = -C_0\,E\,p_1(x),$$

$$-\frac{\alpha\sin 2\gamma}{4\sin(\epsilon+\gamma)}(E^2-1)\,U_1(x) + \left[\frac{\alpha\cos^2\gamma}{2\sin(\epsilon+\gamma)}(E-1)^2 - 2\sin^2\epsilon\,E\right]U_2(x) \qquad (42\,\text{a})$$

$$+\sin\epsilon\cos(\epsilon+\gamma)\,E\,(E-1)\,V_1(x) + \sin\epsilon\sin(\epsilon+\gamma)\,E\,(E+1)\,V_2(x) = -C_0\,E\,p_2(x),$$

$$\cos\epsilon\cos(\epsilon+\gamma)\,(E+1)\,U_1(x) - \sin\epsilon\cos(\epsilon+\gamma)\,(E-1)\,U_2(x) - \left[\frac{\alpha'\sin^2\gamma}{2\sin\epsilon}(E+1)^2\right.$$

$$\left. + 2\cos^2(\epsilon+\gamma)\,E\right]V_1(x) + \frac{\alpha'\sin 2\gamma}{4\sin\epsilon}(E^2-1)\,V_2(x) = -C_0\,E\,q_1(x), \qquad (43\,\text{a})$$

$$-\cos\epsilon\sin(\epsilon+\gamma)\,(E-1)\,U_1(x) + \sin\epsilon\sin(\epsilon+\gamma)\,(E+1)\,U_2(x)$$

$$-\frac{\alpha'\sin 2\gamma}{4\sin\epsilon}(E^2-1)\,V_1(x) + \left[\frac{\alpha'\cos^2\gamma}{2\sin\epsilon}(E-1)^2\right. \qquad (44\,\text{a})$$

$$\left. - 2\sin^2(\epsilon+\gamma)\,E\right]V_2(x) = -C_0\,E\,q_2(x),$$

where $\alpha = Q_1/Q$, $\alpha' = Q_4/Q$ and $C_0 = PL/Q$.

Independent Equations

The operator determinant for the system (41 a) thru (44 a) is

$-\dfrac{\alpha\,\alpha'\cos^2\epsilon\cos^2(\epsilon+\gamma)}{4\sin\epsilon\sin(\epsilon+\gamma)}\,E\,(E-1)^2\,[E^2-2\cos2\gamma\,E+1]^2$ (as compared with $\varDelta^6$ for the straight truss). Therefore, six independent summation constants will appear in the solutions for the following equations for the four deformation components in independent form. The loading is specialized somewhat to an arbitrarily placed joint load on the outer joint at $x=a$. The impulse load has a radial component P and a tangential component equal to sP. Therefore, the load shape functions are $q_1(x)=q_2(x)=0$, $p_1(x)=\delta_x^a$ and $p_2(x)=s\,\delta_x^a$.

$$(E-1)(E^2-2\bar\gamma\,E+1)^2\,U_1(x) = C_0\,(E-1)\,[(D_\gamma\sin^2\epsilon-C_\gamma\cos^2\gamma)\,(E^2-2\bar\gamma\,E+1)$$
$$+F_\gamma\cos\gamma\,E]\,\delta_x^{a-1}+C_0\,(E+1)\,[(C_\gamma\sin\gamma\cos\gamma-D_\gamma\sin\epsilon\cos\epsilon)\,(E^2-2\bar\gamma\,E+1)$$
$$-F_\gamma\sin\gamma\,E]\,s\,\delta_x^{a-1}, \tag{45}$$

$$(E-1)^2(E^2-2\bar\gamma\,E+1)^2\,U_2(x) = C_0\,(E^2-1)\,[(D_\gamma\sin\epsilon\cos\epsilon$$
$$-C_\gamma\sin\gamma\cos\gamma)\,(E^2-2\bar\gamma\,E+1)+F_\gamma\sin\gamma\,E]\,\delta_x^{a-1}$$
$$+C_0\left[(C_\gamma\sin^2\gamma-D_\gamma\cos^2\epsilon)\,(E+1)^2\,(E^2-2\bar\gamma\,E+1)\right. \tag{46}$$
$$\left.-F_\gamma\frac{\sin^2\gamma}{\cos\gamma}\,E\,(E+1)^2+\frac{8}{\alpha}\sin(\epsilon+\gamma)\,E\,(E^2-2\bar\gamma\,E+1)\right]\,s\,\delta_x^{a-1},$$

$$(E-1)(E^2-2\bar\gamma\,E+1)^2\,V_1(x) = C_0\,(E^2-1)\left[\frac{-\cos^2\gamma}{\cos\epsilon\cos(\epsilon+\gamma)}\,(E^2-2\bar\gamma\,E+1)\right.$$
$$\left.+\tfrac{1}{2}F_\gamma\,E\right]\delta_x^a+C_0\left[\frac{\sin\gamma\cos\gamma}{\cos\epsilon\cos(\epsilon+\gamma)}\,(E+1)^2\,(E^2-2\bar\gamma\,E+1)\right. \tag{47}$$
$$\left.-\frac{\tan\gamma}{2}\,F_\gamma\,E\,(E+1)^2-\frac{4\sin\epsilon\sin(\epsilon+\gamma)}{\alpha\cos(\epsilon+\gamma)}\,(E^2-2\bar\gamma\,E+1)\,E\right]\,s\,\delta_x^a,$$

$$(E-1)^2(E^2-2\bar\gamma\,E+1)^2\,V_2(x) = C_0\,(E-1)\left[\frac{-\sin\gamma\cos\gamma}{\cos\epsilon\sin(\epsilon+\gamma)}\,(E+1)^2\,(E^2-2\bar\gamma\,E+1)\right.$$
$$\left.+\frac{\tan\gamma}{2}\,F_\gamma\,E\,(E+1)^2-\frac{4}{\alpha'}\tan\epsilon\sin(\epsilon+\gamma)\,E\,(E^2-2\bar\gamma\,E+1)\right]\delta_x^a$$
$$+C_0\,(E+1)\left[\frac{\sin^2\gamma\,(E+1)^2}{\cos\epsilon\cos(\epsilon+1)}\,(E^2-2\bar\gamma\,E+1)\right. \tag{48}$$
$$\left.-2\,F_\gamma\sin^2\gamma\,E^2-F_\gamma'\,E\,(E^2-2\bar\gamma\,E+1)\right]\,s\,\delta_x^a,$$

where $\quad\bar\gamma=\cos2\gamma;\qquad C_\gamma=\dfrac{2\cos(\epsilon+2\gamma)}{\cos\epsilon\cos^2(\epsilon+\gamma)};\qquad D_\gamma=\dfrac{2\sin(\epsilon+\gamma)}{\alpha\cos^2(\epsilon+\gamma)};$

$$F_\gamma=\frac{2\cos\gamma}{[\cos\epsilon\cos(\epsilon+\gamma)]^2}\left[\frac{\sin(\epsilon+\gamma)}{\alpha}\sin^2 2\epsilon+\sin^2 2\gamma+\frac{\sin\epsilon}{\alpha'}\sin^2(2\epsilon+2\gamma)\right]\quad\text{and}$$

$$F_\gamma'=\frac{2\sin\gamma}{[\cos\epsilon\cos(\epsilon+\gamma)]^2}\left[\frac{\sin(\epsilon+\gamma)}{\alpha}\cos^2\epsilon\sin 2\epsilon+\sin^2\gamma\sin 2\gamma\right.$$
$$\left.-\frac{\sin\epsilon}{\alpha'}\cos^2(\epsilon+\gamma)\sin(2\epsilon+2\gamma)\right].$$

General Solutions

By employing the formulas presented in Appendix A, one can write the general solutions for Eqs. (45) thru (48) with a large number of summation constants as follows:

$$U_1(x) = C_1 \sin 2\gamma x + C_2 x \sin 2\gamma x + C_3 \cos 2\gamma x$$
$$+ C_4 x \cos 2\gamma x + C_5 + U_{1p}(x), \tag{45a}$$

$$U_2(x) = C_6 \sin 2\gamma x + C_7 x \sin 2\gamma x + C_8 \cos 2\gamma x$$
$$+ C_9 x \cos 2\gamma x + C_{10} x + C_{11} + U_{2p}(x), \tag{46a}$$

$$V_1(x) = C_{12} \sin 2\gamma x + C_{13} x \sin 2\gamma x + C_{14} \cos 2\gamma x$$
$$+ C_{15} x \cos 2\gamma x + C_{16} + V_{1p}(x) \tag{47a}$$

$$V_2(x) = C_{17} \sin 2\gamma x + C_{18} x \sin 2\gamma x + C_{19} \cos 2\gamma x$$
$$+ C_{20} x \cos 2\gamma x + C_{21} x + C_{22} + V_{2p}(x), \tag{48a}$$

where the final terms with the p subscript denote the particular sums. Expressions for these terms are obtained by use of the formulas in Appendix B and carrying out routine operations.

$$U_{1p}(x) = \tfrac{1}{4} C_0 \left\{ \left[b_1 s + \frac{F_\gamma(x-a)}{\cos\gamma} \right] \frac{\cos 2\gamma(x-a) - 1}{\cos 2\gamma - 1} - \frac{F_\gamma(x-a)}{\sin\gamma \sin 2\gamma} \right.$$
$$\left. + \left[b_2 + \frac{F_\gamma(x-a)s}{\sin\gamma} \right] \frac{\sin 2\gamma(x-a)}{\sin 2\gamma} \right\} U(x-a), \tag{45b}$$

where $\quad b_1 = 2 C_\gamma \sin 2\gamma - 2 D_\gamma \sin 2\epsilon - \dfrac{F_\gamma}{\sin\gamma}$

and $\quad b_2 = 4 D_\gamma \sin^2\epsilon - 4 C_\gamma \cos^2\gamma + \dfrac{\cos 2\gamma F_\gamma}{\sin\gamma \sin 2\gamma}.$

$$U_{2p}(x) = \tfrac{1}{4} C_0 \left\{ \left[-b_1 + \frac{F_\gamma(x-a)s}{\cos\gamma} \right] \frac{\cos 2\gamma(x-a) - 1}{\cos 2\gamma - 1} + b_3(x-a)s \right.$$
$$\left. - \left[b_4 s + \frac{F_\gamma(x-a)}{\sin\gamma} \right] \frac{\sin 2\gamma(x-a)}{\sin 2\gamma} \right\} U(x-a), \tag{46b}$$

where $\quad b_3 = \dfrac{1}{\sin^2\gamma} \left[4 C_\gamma \sin^2\gamma - 4 D_\gamma \cos^2\epsilon + \dfrac{8}{\alpha}\sin(\epsilon+\gamma) - \dfrac{3}{2}\dfrac{F_\gamma}{\cos\gamma} \right]$

and $\quad b_4 = \dfrac{\cos^2\gamma}{\sin^2\gamma} \left[4 C_\gamma \sin^2\gamma - 4 D_\gamma \cos^2\gamma + \dfrac{8}{\alpha}\dfrac{\sin(\epsilon+\gamma)}{\cos^2\gamma} - \dfrac{F_\gamma}{\cos\gamma} \right].$

$$V_{1p}(x) = \tfrac{1}{4} C_0 \left\{ \left[b_5 s - \frac{2\sin^2 2\gamma}{\cos\epsilon \cos(\epsilon+\gamma)} + F_\gamma(1 + s\tan\gamma)(x-a) \right] \frac{\cos 2\gamma(x-a) - 1}{\cos 2\gamma - 1} \right.$$
$$- \left[b_6 + b_5 s + \frac{F_\gamma s}{\sin 2\gamma} + F_\gamma(1 - s\cot\gamma)(x-a) \right] \frac{\sin 2\gamma(x-a)}{\sin 2\gamma}$$
$$\left. - \frac{F_\gamma(x-a)}{2\sin^2\gamma} + 2\frac{2\cos^2\gamma + s\sin 2\gamma}{\cos\epsilon \cos(\epsilon+\gamma)} \right\} U(x-a), \tag{47b}$$

where $\qquad b_5 = \dfrac{4\cos^2\gamma\sin 2\gamma}{\cos\epsilon\cos(\epsilon+\gamma)} - \dfrac{8\sin\epsilon\sin(\epsilon+\gamma)}{\alpha\cos(\epsilon+\gamma)} - \dfrac{2\,F_\gamma}{\sin 2\gamma}$

and $\qquad b_6 = \dfrac{8\cos^4\gamma}{\cos\epsilon\cos(\epsilon+\gamma)} - \dfrac{\cot\gamma\,F_\gamma}{\sin 2\gamma}.$

$$V_{2p}(x) = \tfrac{1}{4}C_0\left\{[b_7 + b_8\,s + F_\gamma\,(s - \tan\gamma)\,(x-a)]\,\frac{\cos 2\gamma\,(x-a) - 1}{\cos 2\gamma - 1} - \frac{2\sin 2\gamma}{\cos\epsilon\cos(s+\gamma)}\right.$$

$$- \left[b_7 - \cot^2\gamma\,b_8\,s - \frac{F_\gamma\,(\sin\gamma + s\cos\gamma)}{\sin\gamma\sin 2\gamma}\right.$$

$$\left. + F_\gamma\,(1 + s\tan\gamma)\,(x-a)\right]\frac{\sin 2\gamma\,(x-a)}{\sin 2\gamma}$$

$$\left. + \left(b_9\,s + \frac{F_\gamma}{\sin 2\gamma}\,(x-a)\right)\right\}U\,(x-a),$$

(48 b)

where $\qquad b_7 = \dfrac{2\,F_\gamma}{\sin 2\gamma} - \dfrac{4\cos^2\gamma\sin 2\gamma}{\cos\epsilon\cos(\epsilon+\gamma)} - \dfrac{8\sin\epsilon\sin(\epsilon+\gamma)}{\alpha'\cos\epsilon},$

$$b_8 = 2\,F_\gamma' + F_\gamma - \frac{2\sin^2 2\gamma}{\cos\epsilon\cos(\epsilon+\gamma)}$$

and $\qquad b_9 = \dfrac{8}{\cos\epsilon\cos(\epsilon+\gamma)} - \dfrac{3\,F_\gamma}{2\sin^2\gamma} - \dfrac{2\,F_\gamma'}{\sin^2\gamma}.$

As in the preceding problems, the number of summation constants appearing in the general solution is reduced to the correct number of independent quantities by substituting the homogenous portion of the solution back into the dependent system (41 a) thru (44 a). The loading terms are ignored as coefficients of identical functions are matched to give the sixteen relations:

$$C_{21} = \frac{\sin\epsilon}{\sin(\epsilon+\gamma)}C_{10}, \quad C_{22} = \frac{\sin\epsilon}{\sin(\epsilon+\gamma)}(C_{11} - \tfrac{1}{2}C_{10}), \quad C_2 = -C_9, \quad C_4 = C_7,$$

$$C_1 = -C_6, \quad C_{13} = \sin\gamma\,C_7 - \cos\gamma\,C_9, \quad C_{15} = \sin\gamma\,C_7 + \sin\gamma\,C_9,$$

$$C_{18} = \cos\gamma + \sin\gamma\,C_9, \quad C_{20} = -\sin\gamma\,C_7 + \cos\gamma\,C_9, \quad C_3 = C_8,$$

$$C_5 = \frac{\cot\gamma}{2}\,\frac{\dfrac{\sin\epsilon}{\alpha\cos\gamma}\cos^2\epsilon + \sin^2\gamma + \dfrac{\sin\epsilon}{\alpha'}\cos^2(\epsilon+\gamma)}{\dfrac{\sin(\epsilon+\gamma)}{\alpha}\cos^2\epsilon + \sin^2\gamma + \dfrac{\sin\epsilon}{\alpha'}\cos^2(\epsilon+\gamma)}\,C_{10},$$

$$C_{16} = \frac{\cot\gamma\sin\epsilon}{2\sin(\epsilon+\gamma)}\,\frac{\dfrac{\sin(\epsilon+\gamma)}{\alpha}\cos^2\epsilon + \sin^2\gamma + \dfrac{\sin(\epsilon+\gamma)}{\alpha'\cos\gamma}\cos^2(\epsilon+\gamma)}{\dfrac{\sin(\epsilon+\gamma)}{\alpha}\cos^2\epsilon + \sin^2\gamma + \dfrac{\sin\epsilon}{\alpha'}\cos^2(\epsilon+\gamma)}\,C_{10},$$

$$C_{12} = \sin\gamma\,C_6 - \cos\gamma\,C_8 - e_1\,C_7 + e_2\,C_9, \quad C_{14} = \cos\gamma\,C_6 + \sin\gamma\,C_8 - e_2\,C_7 - e_1\,C_9,$$

$$C_{17} = \cos\gamma\,C_6 + \sin\gamma\,C_8 - e_2\,C_7 - e_1\,C_9, \quad C_{19} = -\sin\gamma\,C_6 + \cos\gamma\,C_8 + e_1\,C_7 - e_2\,C_9.$$

where $\qquad e_1 = \dfrac{1}{4\cos^2(\epsilon+\gamma)}\left[\dfrac{\sin 2\gamma}{\sin\epsilon}\cos\gamma - 2\cos(\epsilon+\gamma)\sin(\epsilon+2\gamma)\right]$

and $\qquad e_2 = \dfrac{1}{4\cos^2(\epsilon+\gamma)}\left[\dfrac{\sin 2\gamma}{\sin\epsilon}\sin\gamma - 2\cos(\epsilon+\gamma)\cos(\epsilon+2\gamma)\right].$

Boundary Conditions

The six independent summation constants can be evaluated for any specific problem by substituting the appropriate boundary conditions into the general solutions. No one set of boundary conditions seems likely to enjoy usage sufficient enough to warrant the derivation here of a set of formulas for the corresponding constants. Two reasonable sets which do not employ special structural members are discussed briefly here to illustrate typical relations needed to determine the constants.

The two hinged arched truss supported at the inner terminal joints only presents a plausible set of boundary conditions. The relations needed to determine the six constants are:

$$V_1(0) = V_2(0) = V_1(n) = V_2(n) = S_1(0) = S_1(n) = 0. \tag{49}$$

For a fixed truss with supports at the outer, as well as the inner, terminal joints, the first four conditions are unchanged but the last two are replaced as follows:

$\overline{U}(-1)\overline{l}_1 = \overline{U}(n)\overline{l}_1 = 0$, or in scalar form the second of these conditions is

$$(\sin\gamma)\,U_1(n) + (\cos\gamma)\,U_2(n) = 0. \tag{49a}$$

Other boundary conditions can be expressed in a similar manner resulting in equations solvable for the summation constants needed to produce working formulas from the general solutions.

The Circular Vierendeel Truss

The circular Vierendeel truss, though not as popular as the circular triangulated truss, also represents a system which is used frequently by the structural engineer but has not been analyzed in a closed form suitable for
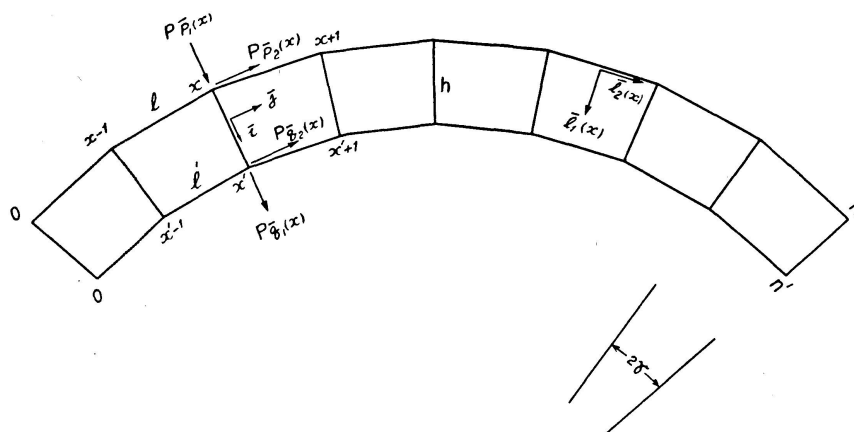

Fig. 5. Circular Vierendeel Truss.

design studies. The general closed form solution for stresses and deflections resulting from an arbitrary loading on the circularly arched Vierendeel truss is the goal of this section. Such a solution would be helpful in the analysis of a cylindrical space lattice having the circular Vierendeel as a typical transverse rib.

Force-Deformation Relations

The approach will again be that of using the joint deformations as unknowns. Each joint has three deformation components; rotation, radial movement, and tangential movement (see Fig. 5) Then the joint moments can be written:

$$M_{x,x+1} = E\,M_{x-1,x} = 2\,K_m\left[(E+2)\,\theta_x - \frac{3}{L}\,(E-1)\,\overline{U}\,(x)\,\bar{l}_1\right];\qquad(50)$$

$$M_{x+1,x} = E\,M_{x,x-1} = 2\,K_m\left[(2\,E+1)\,\theta_x - \frac{3}{L}\,(E-1)\,\overline{U}\,(x)\,\bar{l}_1\right];\qquad(51)$$

$$M_{x,x'} = 2\,K_c\left[2\,\theta_x + \theta_x' - \frac{3}{h}\,\overline{U}\,(x) - \overline{U}'\,(x)\bar{j}\right];\qquad(52)$$

$$M_{x',x} = 2\,K_c\left[\theta_x + 2\,\theta_x' - \frac{3}{h}\,\overline{U}\,(x) - \overline{U}'\,(x)\bar{j}\right];\qquad(53)$$

and equations identical to (50) and (51) for the deformation components of the inner joints i.e., the primed (') quantities; where θ_x = rotation of outer joint x, $\overline{U}\,(x)$ = vector movement of outer joint x equal to $U_1(x)\bar{i} + U_2(x)\bar{j}$, K_m and K_c = moment stiffness of outer main and cross members respectively, L and h = length of outer main and cross members respectively, $\bar{l}_1$ and $\bar{l}_2$ = unit vectors perpendicular and parallel respectively to the main chord members.

Compatibility Relations

In addition to the usual compatibility assumption of rigid joints i.e., θ's equal for all members connected at the joint, the effect of axial deformations upon the joint movements will be disregarded.

$$U_1(x) = U_1'(x),\qquad(54)$$

$$(E-1)\,\overline{U}\,(x)\bar{l}_2 = 0,\qquad(55)$$

$$(E-1)\,\overline{U}'\,(x)\bar{l}_2 = 0.\qquad(55')$$

Carrying out the indicated operations in (55) and (55'), one gets the following scalar equations:

$$(E-1)\,U_2(x) = \tan\gamma\,(E+1)\,U_1(x),\qquad (E-1)\,U_2'(x) = \tan\gamma\,(E+1)\,U_1'(x)$$

or
$$(E-1)\,[U_2(x) - U_2'(x)] = 0.$$

With these relations, the second scalar term in Eqs. (50) and (51) can be written in a useful form.

$$(E-1)\,\overline{U}\,(x)\overline{l}_1 = (E-1)\,\overline{U}'\,(x)\overline{l}_1 = \frac{\varDelta^{-1}}{\cos\gamma}\,(E^2 - 2\cos 2\gamma\,E + 1)\,U_1\,(x).$$

Equilibrium Equations

Summation of moments at the outer and inner joints provide two of the three equations needed to solve for the unknown joint deformation components θ_x, θ'_x and $U_1(x)$. These equations, after term by term operation with $E^2 - E$, are:

$$(E-1)\,[E^2 + 2\,(2+\alpha)\,E + 1]\,\theta_x + \alpha\,(E^2 - E)\,\theta'_x$$
$$-3\,(E+1)\,(E^2 - 2\,\bar{\gamma}\,E + 1)\,U_x = 0, \tag{56}$$

$$\alpha'\,(E^2 - E)\,\theta_x + (E-1)\,[E^2 + 2\,(2+\alpha')\,E + 1]\,\theta'_x$$
$$-3\,r\,(E+1)\,(E^2 - 2\,\bar{\gamma}\,E + 1)\,U_x = 0, \tag{57}$$

where $\quad \alpha = \dfrac{K_c}{K_m}, \quad \alpha' = \dfrac{K_c}{K'_m}, \quad r = \dfrac{L}{L'}, \quad \bar{\gamma} = \cos 2\gamma \quad$ and $\quad U_x = \dfrac{U_1\,(x)}{L\cos\gamma}.$

The third equation is obtained by summing radial forces on the cross member and tangential forces on the inner and outer joints of element x. Combining the resultant equations so as to eliminate the unknown thrusts in the outer and inner chord members, one obtains the third equation needed to analyse the system. After the introduction of impulse load functions,

$$\overline{p}_1\,(x) + \overline{q}_1\,(x) = \delta_x^a \quad \text{and} \quad \overline{p}_2\,(x) + \overline{q}_2\,(x) = s\,\delta_x^a,$$

this equation is written:

$$(E^2 - 2\,\bar{\gamma}\,E + 1)\left[(E^2 - 1)\,\theta_x + \frac{r\,\alpha}{\alpha'}\,(E^2 - 1)\,\theta'_x - 2\left(1 + r^2\frac{\alpha}{\alpha'}\right)(E^2 - 2\,\bar{\gamma}\,E + 1)\,U_x\right] =$$
$$\frac{PL}{6\,K_m}[\cos\gamma\,\varDelta^2 - \sin\gamma\,(E^2 - 1)\,s]\,\delta_x^{a-1}. \tag{58}$$

General Solutions

The operator determinant for the dependent Eqs. (56), (57), and (58), after cancellation of $\varDelta^2$ common to all terms of the independent equations, is of degree eight signifying the fact that the general solution will contain eight independent summation constants to be determined by conditions at the boundaries. It may be written:

$$\left(1 + r^2 \frac{\alpha}{\alpha'}\right) (E^2 - 2\,\bar{\gamma}\,E + 1)^2 \,(E^4 - 2\,g\,E^3 - 2\,h\,E^2 - 2\,g\,E + 1),$$

where $\quad g = \dfrac{1}{1 + r^2 \frac{\alpha}{\alpha'}}\left[r\,\alpha\,(3 + 2\,r) - r^2 \dfrac{\alpha}{\alpha'}\,(1 + \alpha) + 2\,\alpha - (1 + \alpha') + 2\,\alpha\right], \quad \bar{\gamma} = \cos 2\gamma$

and $\quad h = \dfrac{1}{1 + r^2 \frac{\alpha}{\alpha'}}\left[2\,r\,\alpha\,(3 + 4\,r + r\,\alpha) + r^2 \dfrac{\alpha}{\alpha'}\,(3 + 2\,\alpha) + \alpha'\,(2 + 3\,\alpha) + 3 + 8\,\alpha\right]$

or letting $\quad f(E) = (E^2 - 2\,\bar{\gamma}\,E + 1)\,(E^2 - 2\,j\,E + 1)\,(E^2 + 2\,k\,E + 1),$

where $\qquad\qquad\qquad j = \tfrac{1}{2}\,(g + \sqrt{g^2 + 2\,h + 2})$

and $\qquad\qquad\qquad k = \tfrac{1}{2}\,(\sqrt{g^2 + 2\,h + 2} - g)$

the operator determinant may be written:

$$\left(1 + r^2 \frac{\alpha}{\alpha'}\right)(E^2 - 2\,\bar{\gamma}\,E + 1)\,f(E).$$

The governing equations can now be written in independent form:

$$f(E)\,\theta_x = 3\,C_0\,(E + 1)\,[E^2 + 2\,(D_\alpha)\,E + 1]\,[\cos\gamma\,(E - 1) - \sin\gamma\,(E + 1)s]\delta_x^{a-1}, \quad (56\,\text{a})$$

$$f(E)\,\theta'_x = 3\,r\,C_0\,(E + 1)\,[E^2 + 2\,(D'_\alpha)\,E + 1]\,[\cos\gamma\,(E - 1) - \sin\gamma\,(E + 1)s]\delta_x^{a-1}, \quad (57\,\text{a})$$

$$(E^2 - 2\,\bar{\gamma}\,E + 1)\,f(E)\,U_x = \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (58\,\text{a})$$

$$C_0\,(E - 1)\,(E^2 + 2\,C_\alpha\,E + 1)\,(E^2 + 2\,C'_\alpha\,E + 1)\,[\cos\gamma\,(E - 1) - \sin\gamma\,(E + 1)s]\,\delta_x^{a-1},$$

where $\quad C_0 = \dfrac{1}{1 + r^2 \frac{\alpha}{\alpha'}}\,\dfrac{P\,L}{6\,K_m}; \qquad C_\alpha, C'_\alpha = \tfrac{1}{2}\,[4 + \alpha + \alpha' \pm \sqrt{\alpha^2 - \alpha\,\alpha' + \alpha'^2}];$

$$D_\alpha = 2 + \alpha' - \frac{\alpha\,r}{2}; \qquad D'_\alpha = 2 + \alpha - \frac{\alpha'}{2\,r}.$$

Now, the general solution, in terms of excess summation constants is:

$$U_x = U_{xp} + (C_1 + C_2\,x)\cos 2\gamma\,x + (C_3 + C_4\,x)\sin 2\gamma\,x + C_5 \cosh \varphi\,x + C_6 \sinh \varphi\,x$$
$$+ (-1)^x (C_7 \cosh \eta\,x + C_8 \sinh \eta\,x), \qquad\qquad\qquad (58\,\text{b})$$

$$\theta_x = \theta_{xp} + C_9 \cos 2\gamma\,x + C_{10} \sin 2\gamma\,x + C_{11} \cosh \varphi\,x + C_{12} \sinh \varphi\,x$$
$$+ (-1)^x (C_{13} \cosh \eta\,x + C_{14} \sinh \eta\,x), \qquad\qquad\qquad (56\,\text{b})$$

$$\theta'_x = \theta'_{xp} + C_{15} \cos 2\gamma\,x + C_{16} \sin 2\gamma\,x + C_{17} \cosh \varphi\,x + C_{18} \sinh \varphi\,x$$
$$+ (-1)^x (C_{19} \cosh \eta\,x + C_{20} \sinh \eta\,x), \qquad\qquad\qquad (57\,\text{b})$$

where $\cosh \varphi = j$, $\cosh \eta = k$, and terms with subscript p denote particular solutions to the corresponding independent equation.

$$\theta_{xp} = 3\,C_0\,\frac{(j+1)\,(j+D_\alpha)}{(j+k)\,(j-\bar\gamma)}\left[\cos\gamma\,\frac{\cosh\varphi\,(x-a)}{\cosh\varphi+1}-\sin\gamma\,\frac{\sinh\varphi\,(x-a)}{\sinh\varphi}s\right]U\,(x-a)$$

$$-3\,C_0\,\frac{(-1)^{x-a}\,(k-1)\,(k-D_\alpha)}{(k+j)\,(k+\bar\gamma)}\left[\cos\gamma\,\frac{\cosh\eta\,(x-a)}{\cosh\eta-1}\right.$$

$$\left.-\sin\gamma\,\frac{\sinh\eta\,(x-a)}{\sinh\eta}s\right]U\,(x-a)+3\,C_0\,\frac{(\bar\gamma-1)\,(\bar\gamma+D_\alpha)}{(\bar\gamma-j)\,(\bar\gamma+k)}\left[\cos\gamma\,\frac{\cos 2\,\gamma\,(x-a)}{\cos 2\,\gamma+1}\right.$$

$$\left.-\sin\gamma\,\frac{\sin 2\,\gamma\,(x-a)}{\sin 2\,\gamma}s\right]U\,(x-a)\,,$$

(56 c)

$$\theta'_{xp} = [\text{same as } \theta_{xp} \text{ except } 3\,C_0 \text{ is replaced by } 3\,r\,C_0 \text{ and } D_\alpha \text{ by } D'_\alpha]\,,$$

$$U_{xp} = \frac{C_0\,U\,(x-a)}{4\,(j+k)\,(j-\bar\gamma)^2}\,\alpha_j\left[\cos\gamma\,\frac{\sinh\varphi\,(x-a)}{\sinh\varphi}-\sin\gamma\,\frac{\cosh\varphi\,(x-a)}{\cosh\varphi-1}s\right]$$

$$+\frac{C_0\,U\,(x-a)}{4\,(k+j)\,(k+\bar\gamma)^2}\,(-1)^{x-a}\,\alpha_{-k}\left[\cos\gamma\,\frac{\sinh\eta\,(x-a)}{\sinh\eta}-\sin\gamma\,\frac{\cosh\eta\,(x-a)}{\cosh\eta+1}s\right]$$

$$+\frac{C_0\,U\,(x-a)}{8\,(\bar\gamma-j)(\bar\gamma+k)(1-\bar\gamma^2)}\,\alpha_{\bar\gamma}\left\{\left[d_0\,s-(1+3\,\bar\gamma)\,s\right.\right.$$

$$\left.+\sin 2\,\gamma\,[2\,(x-a)+7]\right]\frac{\cos 2\,\gamma\,(x-a)}{\cos 2\,\gamma-1}\sin\gamma$$

$$+\left[-d_0+(1+3\,\gamma)+\sin 2\,\gamma\,[2\,(x-a)+7]\right]\frac{\sin 2\,\gamma\,(x-a)}{\sin 2\,\gamma}\cos\gamma\right\}$$

(58 c)

$$+\frac{C_0\,U\,(x-a)}{8\,(\bar\gamma-j)\,(\bar\gamma+k)\,(1-\bar\gamma^2)}\left\{[d_1\,s\sin\gamma+d_2\cos\gamma]\cos 2\,\gamma\,(x-a)\right.$$

$$+[d_3\,s\sin\gamma+d_4\cos\gamma]\frac{\sin 2\,\gamma\,(x-a)}{\sin 2\,\gamma}\bigg\}+\left\{\sin\gamma\,\frac{(8\,j^3-4\,j^2-4\,j+1)}{4\,(j-1)\,(j+k)\,(j-\bar\gamma)^2}s\right.$$

$$-\sin\gamma\,\frac{(8\,k^3+4\,k^2-4\,k-1)}{4\,(k+1)\,(k+j)\,(k+\bar\gamma)^2}s-\sin\gamma\,\frac{(8\,\bar\gamma^3-4\,\bar\gamma^2-4\,\gamma+1)}{8\,(\bar\gamma-j)\,(\bar\gamma+k)\,(1-\bar\gamma^2)}(d_0+1)\,s$$

$$-\frac{(8\,\bar\gamma^3+4\,\bar\gamma^2-4\,\bar\gamma-1)}{2\,(\bar\gamma-j)\,(\bar\gamma+k)\,(1+\bar\gamma)}-\frac{3\,(4\,\bar\gamma^2-2\,\bar\gamma-1)}{8\,(\bar\gamma-j)\,(\bar\gamma+k)\,(1-\bar\gamma^2)}\bigg\}C_0\,U\,(x-a)\,,$$

where

$$\alpha_z = (4\,z^3-4\,z+1)+(z-1)\,[4\,(3+\alpha+\alpha')\,z+4\,(\alpha+2)\,(\alpha'+2)-\alpha\,\alpha']\,,$$

$$d_0 = 2\,(1-\bar\gamma^2)\,\frac{(2\,\bar\gamma-j-k)}{(\bar\gamma-j)\,(\bar\gamma+k)}\,,$$

$$d_1 = 2\,(\alpha+\alpha')\,(64\,\bar\gamma^6-208\,\bar\gamma^5-1040\,\bar\gamma^4-428\,\bar\gamma^3+562\,\bar\gamma^2+253\,\bar\gamma-32)$$

$$-3\,\alpha\,\alpha'\,(32\,\bar\gamma^5+232\,\bar\gamma^4+124\,\bar\gamma^3+144\,\bar\gamma^2-63\,\bar\gamma+10)+(864\,\bar\gamma^6-1088\,\bar\gamma^5$$

$$-4408\,\bar\gamma^4-228\,\bar\gamma^3+2146\,\bar\gamma-99)\,,$$

$$d_2 = -2\,(\alpha+\alpha')\,(112\,\bar\gamma^5+736\,\bar\gamma^4+228\,\bar\gamma^3-694\,\bar\gamma^2-213\,\bar\gamma+29)-6\,\alpha\,\alpha'\,(80\,\bar\gamma^4$$

$$+40\,\bar\gamma^3-80\,\bar\gamma^2-23\,\bar\gamma+3)+(336\,\bar\gamma^5+1744\,\bar\gamma^4+1664\,\bar\gamma^3-1992\,\bar\gamma^2-868\,\bar\gamma+469)\,,$$

$$d_3 = 2\,(\alpha + \alpha')\,(128\,\bar{\gamma}^7 + 368\,\bar{\gamma}^6 - 176\,\bar{\gamma}^5 - 340\,\bar{\gamma}^4 + 338\,\bar{\gamma}^3 + 97\,\bar{\gamma}^2 - 219\,\bar{\gamma} - 54)$$

$$- 3\,\alpha\,\alpha'\,(32\,\bar{\gamma}^6 + 232\,\bar{\gamma}^5 + 108\,\bar{\gamma}^4 - 260\,\bar{\gamma}^3 - 135\,\bar{\gamma}^2 + 55\,\bar{\gamma} + 22)$$

and $+ (864\,\bar{\gamma}^7 + 1952\,\bar{\gamma}^6 - 696\,\bar{\gamma}^5 - 1580\,\bar{\gamma}^4 - 1436\,\bar{\gamma}^3 + 188\,\bar{\gamma}^2 - 855\,\bar{\gamma} - 193)$,

$$d_4 = -4\,(\alpha + \alpha')\,(136\,\bar{\gamma}^6 + 348\,\bar{\gamma}^5 - 64\,\bar{\gamma}^4 - 515\,\bar{\gamma}^3 - 107\,\bar{\gamma}^2 + 167\,\bar{\gamma} + 35)$$

$$- 60\,\alpha\,\alpha'\,(\bar{\gamma}^2 - 1)\,(8\,\bar{\gamma}^3 + 4\,\bar{\gamma}^2 - 4\,\bar{\gamma} - 1) - (1632\,\bar{\gamma}^6 + 2976\,\bar{\gamma}^5 - 1416\,\bar{\gamma}^4 - 4400\,\bar{\gamma}^3$$

$$- 476\,\bar{\gamma} + 259).$$

As indicated previously, the general solutions (56 b), (57 b), and (58 b) contain an excess of summation constants due to the fact that the system (56), (57), and (58) is of order eight. The necessary relations to reduce the constants to eight independent quantities are obtained by substitution of the general solution into the original system and matching like coefficients. Selecting C_1 thru C_8 as independent constants, the results are:

$$\frac{C_9}{C_2} = \frac{C_{10}}{C_4} = \frac{6\,(1 + \bar{\gamma})\,[2\,(2 + \bar{\gamma} + \alpha') - r\,\alpha]}{4\,(2 + \bar{\gamma} + \alpha')\,(2 + \bar{\gamma} + \alpha) - \alpha\,\alpha'},$$

$$\frac{C_{15}}{C_2} = \frac{C_{16}}{C_4} = \frac{6\,(1 + \bar{\gamma})\,[2\,r\,(2 + \bar{\gamma} + \alpha) - \alpha']}{4\,(2 + \bar{\gamma} + \alpha')\,(2 + \bar{\gamma} + \alpha) - \alpha\,\alpha'},$$

$$\frac{C_{11}}{C_6} = \frac{C_{12}}{C_5} = \frac{j - \bar{\gamma}}{\sinh \varphi}\,\frac{(r^2\,\alpha + \alpha')\,[2\,(2 + j + \alpha') - r\,\alpha]}{(2 + j)\,(r^2\,\alpha + \alpha') + (r\,\alpha - \alpha')^2 + r\,\alpha\,\alpha'},$$

$$\frac{C_{17}}{C_6} = \frac{C_{18}}{C_5} = \frac{j - \bar{\gamma}}{\sinh \varphi}\,\frac{(r^2\,\alpha + \alpha')\,[2\,r\,(2 + j + \alpha) - \alpha']}{(2 + j)\,(r^2\,\alpha + \alpha^*) + (r\,\alpha - \alpha')^2 + r\,\alpha\,\alpha'},$$

$$\frac{C_{13}}{C_8} = \frac{C_{14}}{C_7} = \frac{k + \bar{\gamma}}{\sinh \eta}\,\frac{(r^2\,\alpha + \alpha')\,[2\,(2 - k + \alpha') - r\,\alpha]}{(2 - k)\,(r^2\,\alpha + \alpha') + (r\,\alpha - \alpha')^2 + r\,\alpha\,\alpha'},$$

$$\frac{C_{19}}{C_8} = \frac{C_{20}}{C_7} = \frac{k + \bar{\gamma}}{\sinh \eta}\,\frac{(r^2\,\alpha + \alpha')\,[2\,r\,(2 - k + \alpha) - \alpha')}{(2 - k)\,(r^2\,\alpha + \alpha') + (r\,\alpha - \alpha')^2 + r\,\alpha\,\alpha'}.$$

Boundary Conditions

The eight independent summation constants C_1 thru C_8 are to be determined by insertion of the boundary conditions into the general solutions. As with the triangulated truss, no one set of boundary conditions seems to warrant the special consideration of having the corresponding constants evaluated here. However, the conditions necessary to compute one reasonable set, corresponding to terminal fixity of both chords, can be easily listed as follows: $U_0 = \theta_0 = \theta_0' = U_2(0) = U_n = \theta_n = \theta_n' = U_2(n) = 0$, where Eq. (55) is used to give $U_2(x)$ in terms of U_x, i. e., $U_2(x) = L \sin \gamma \frac{(E + 1)}{(E - 1)}\,U_x$.

Conclusions

The four lattice solutions presented in this paper should prove useful for design studies of the corresponding structural systems. Since an impulse joint loading was employed throughout, the results can be used for a wide variety of purposes. For example, simple superposition of the results for an arbitrarily placed impulse load i. e., influence function, will give the answers needed to analyse the structure with any static loading condition including intermediate supports. In different terminology, the solutions represent a closed form for the complete flexibility matrix, a quantity frequently used in analysis for dynamic loading.

The cases used to illustrate a closed form approach to lattice analysis were varied and demonstrate the procedure used for most regular lattices for which the mathematical model is a system of ordinary difference equations. The author's experience in this area leads to the prediction that a number of structural lattices, not referred to here, have governing equations identical to the ones solved in this paper. Thus the solutions are probably applicable to numerous lattice systems.

Unless one is alert to the possibilities for algebraic contraction and transformations to maintain symmetry, the expressions resulting from the closed form solution of simultaneous difference equations tend to become long and unwieldy.

Acknowledgements

Appendix A
Solutions for Difference Equations with Symmetric Quartic Operators

The symmetric quartic operator appears frequently in the formulation of governing equations for structural lattices. Solutions for the various cases of a homogeneous difference equation containing such an operator are given below.

Assume an equation of the form

$$(E^4 - 2g\,E^3 + 2h\,E^2 - 2g\,E + 1)\,Y_x = 0, \tag{1}$$

where g and h are real constants with either positive or negative values. The operator can be factored into two symmetric quadratic operators

$$(E^2 - 2jE + 1)(E^2 - 2kE + 1)Y_x = 0, \tag{2}$$

where $\quad j = \tfrac{1}{2}[g + \sqrt{g^2 - 2(h-1)}] \quad$ and $\quad k = \tfrac{1}{2}[g - \sqrt{g^2 - 2(h-1)}].$

Therefore, the general solution of (1) consists of the sum of solutions for two second order equations of special form,

$$(E^2 - 2jE + 1)Y_x' = 0. \tag{2'}$$

The possible cases are:

a) 'j' real and positive, $j^2 > 1$; then

$$Y_x' = C_1 \cosh \varphi x + C_2 \sinh \varphi x, \tag{2' a}$$

where $\pm \varphi = \cosh^{-1} j$ or $\varphi = \log_e (j + \sqrt{j^2 - 1})$. $j^2 = 1$; then

$$Y_x' = C_1 + C_2 x. \tag{2' b}$$

b) 'j' real and negative; $j^2 > 1$, then

$$Y_x' = (-1)^x (C_1 \cosh \varphi x + C_2 \sinh \varphi x), \tag{2' c}$$

where $\pm \varphi = \cosh^{-1}(-j)$ or $\varphi = \log_e(-j + \sqrt{j^2 - 1})$, $j^2 = 1$, then

$$Y_x' = (-1)^x (C_1 + C_2 x). \tag{2' d}$$

c) 'j' real, positive or negative and $j^2 < 1$, then

$$Y_x' = C_1 \cos \varphi x + C_2 \sin \varphi x, \quad \text{where} \quad \pm \varphi = \cos^{-1} j. \tag{2' e}$$

d) 'j' complex. This occurs if $h > 1 + \tfrac{1}{2} g^2$, in which case j and k form a conjugate pair as do $j + \sqrt{j^2 - 1}$ and $k + \sqrt{k^2 - 1}$.

Since the roots of the quadratic operator on Y_x' do not form a conjugate pair, it is more convenient to return to the original quartic operator shown in (2).
    In this case, the solution for (2) can be written

$$Y_x = \rho^x (C_1 \cos \varphi x + C_2 \sin \varphi x) + \rho^{-x} (C_3 \cos \varphi x + C_4 \sin \varphi x), \tag{2 a}$$

where ρ and φ are the amplitude and argument respectively for the polar form of the complex number $j + \sqrt{j^2 - 1}$. The polar form of j is

$$j = \bar{\rho}(\cos \bar{\varphi} + i \sin \bar{\varphi}), \quad \text{where} \quad \bar{\rho} = \tfrac{1}{2}\sqrt{2(h-1)} \quad \text{and} \quad \bar{\varphi} = \cos^{-1}\frac{g}{\sqrt{2(h-1)}}.$$

An alternate form of the above solution, which is more convenient for some boundary conditions, is obtained by regrouping the terms and changing summation constants, as follows:

$$\begin{aligned} Y_x = {}& C_1' \cosh \eta x \cos \varphi x + C_2' \cosh \eta x \sin \varphi x \\ & + C_3' \sinh \eta x \cos \varphi x + C_4' \sinh \eta x \sin \varphi x. \end{aligned} \tag{2 b}$$

Where φ is as defined above and $\eta = \log_e \rho$. Readers conversant with the theory of complex variables will observe the following relations in the transformation to the alternate form:

$$j = \cosh \eta \cos \varphi + i \sinh \eta \sin \varphi, \qquad \sqrt{j^2 - 1} = \sinh \eta \cos \varphi + i \cosh \eta \sin \varphi$$

and

$$\log_e [j + \sqrt{j^2 - 1}] = \eta + i \varphi.$$

It can also be seen that the alternate form for the complex case includes cases a), b), and c) as special forms, i.e. case a) corresponds to $\varphi = 0$, case b) corresponds to $\varphi = \pi$ and case c) corresponds to $\eta = 0$.

Appendix B
Particular Solutions for Difference Equations with Step Function Loading Terms

One of the key problems encountered in analysing a structural lattice with generalized loading is that of solving a difference equation with a finite step function for the inhomogeneous term. Solutions for a number of typical cases follow.

Given $(E - \beta) Y_x = U(x - a)$, then

$$Y_x = \frac{1}{\beta - 1} [\beta^{x-a} - 1] U(x - a). \tag{1}$$

Given $(E - \beta_1)(E - \beta_2) Y_x = U(x - a)$, roots real or complex, then

$$Y_x = [C_1(\beta_1^{x-a} - 1) + C_2(\beta_2^{x-a} - 1)] U(x - a), \tag{2}$$

where $C_1 = [(\beta_1 - 1)(\beta_1 - \beta_2)]^{-1}$ and $C_2 = [(\beta_2 - 1)(\beta_2 - \beta_1)]^{-1}$.

Given $(E - \beta)^2 Y_x = U(x - a)$, a limiting case of (2), then

$$Y_x = \frac{1}{(\beta - 1)^2} \left\{ \left[\frac{\beta - 1}{\beta} (x - a) - 1 \right] \beta^{x-a} + 1 \right\} U(x - a). \tag{2a}$$

Given $(E^2 - 2j E + 1) Y_x = U(x - a)$, a special case of (2), then

$$Y_x = \frac{1}{2(\beta - 1)\sqrt{j^2 - 1}} (\beta^{x-a} + \beta^{a-x+1} - (\beta + 1)] U(x - a), \tag{2b}$$

where $\beta = j + \sqrt{j^2 - 1}$. In many instances it is more convenient to use transcendental functions. For $j > +1$ and real

$$Y_x = \frac{1}{2} \left[\frac{\cosh \eta (x - a) - 1}{\cosh \eta - 1} - \frac{\sinh \eta (x - a)}{\sinh \eta} \right] U(x - a), \tag{2c}$$

where $\cosh \eta = j$ or $\eta = \log_e (j + \sqrt{j^2 - 1})$. Other ranges of values for j result in complex values for η, thus changing the form of (2c). In general,

$\log_e (j + \sqrt{j^2 - 1}) = \eta + i\,\varphi$, where e^η is the amplitude and φ is the argument of the quantity $j + \sqrt{j^2 - 1}$. If $j > +1$, then $\varphi = 0$ giving the results shown in (2c). If $j < -1$, then $\varphi = \pi$. Since $\cosh(\eta + i\pi)x = (-1)^x \cosh \eta x$ and $\sinh(\eta + i\pi)x = (-1)^x \sinh \eta$, the results for $j < -1$ become

$$Y_x = -\frac{1}{2}\left[\frac{(-1)^{x-a}\cosh \eta\,(x-a) - 1}{\cosh \eta + 1} - \frac{(-1)^{x-a}\sinh \eta\,(x-a)}{\sinh \eta}\right] U(x-a), \quad (2\,\mathrm{d})$$

where $\cosh \eta = -j$. If $j^2 < 1$, then $\eta = 0$ and the results become

$$Y_x = \frac{1}{2}\left[\frac{\cos \varphi\,(x-a) - 1}{\cos \varphi - 1} - \frac{\sin \varphi\,(x-a)}{\sin \varphi}\right] U(x-a), \quad \text{where } \cos\varphi = j. \quad (2\,\mathrm{e})$$

Given $(E - \beta_1)(E - \beta_2)(E - \beta_3) Y_x = U(x-a)$, roots real or complex, then

$$Y_x = [C_1(\beta_1^{x-a} - 1) + C_2(\beta_2^{x-a} - 1) + C_3(\beta_3^{x-a} - 1)] U(x-a), \quad (3)$$

where $\quad C_1 = [(\beta_1 - 1)(\beta_1 - \beta_2)(\beta_1 - \beta_3)]^{-1}, \qquad C_2 = [(\beta_2 - 1)(\beta_2 - \beta_1)(\beta_2 - \beta_3)]^{-1},$
and $\qquad C_3 = [(\beta_3 - 1)(\beta_3 - \beta_1)(\beta_3 - \beta_2)]^{-1}.$

Given $(E - \beta_1)(E - \beta_2)(E - \beta_3)(E - \beta_4) Y_x = U(x-a)$, roots real or complex, then

$$Y_x = [C_1(\beta_1^{x-a} - 1) + C_2(\beta_2^{x-a} - 1) + C_3(\beta_3^{x-a} - 1) + C_4(\beta_4^{x-a} - 1)] U(x-a), \quad (4)$$

where $\qquad C_1 = [(\beta_1 - 1)(\beta_1 - \beta_2)(\beta_1 - \beta_3)(\beta_1 - \beta_4)]^{-1},$

$$C_2 = [(\beta_2 - 1)(\beta_2 - \beta_1)(\beta_2 - \beta_3)(\beta_2 - \beta_4)]^{-1},$$

$$C_2 = [(\beta_3 - 1)(\beta_3 - \beta_1)(\beta_3 - \beta_2)(\beta_3 - \beta_4)]^{-1},$$

$$C_4 = [(\beta_4 - 1)(\beta_4 - \beta_1)(\beta_4 - \beta_2)(\beta_4 - \beta_3)]^{-1}.$$

Given $(E^2 - 2j\,E + 1)(E^2 - 2k\,E + 1) Y_x = U(x-a)$, a special case of (4), then

$$Y_x = \frac{1}{2(\beta^2 - 1)(\beta - 1)(j - k)}[\beta^{x-a} + \beta^{a-x+3} - (\beta^3 + 1)] U(x-a)$$

$$+ \frac{1}{2(\gamma^2 - 1)(\gamma - 1)(k - j)}[\gamma^{x-a} + \gamma^{a-x+3} - (\gamma^3 + 1)] U(x-a), \quad (4\,\mathrm{a})$$

where $\beta = j + \sqrt{j^2 - 1}$ and $\gamma = k + \sqrt{k^2 - 1}$. If j and k are real positive and greater than 1, hyperbolic functions may be used

$$Y_x = \frac{1}{4(j - k)}\left[(2j - 1)\frac{\cosh \eta\,(x-a) - 1}{\cosh \eta - 1} - (2j + 1)\frac{\sinh \eta\,(x-a)}{\sinh \eta}\right] U(x-a)$$

$$+ \frac{1}{4(k - j)}\left[(2k - 1)\frac{\cosh \lambda\,(x-a) - 1}{\cosh \lambda - 1} - (2k + 1)\frac{\sinh \lambda\,(x-a)}{\sinh \lambda}\right] U(x-a), \quad (4\,\mathrm{b})$$

where $\eta = \cosh^{-1}(j)$, and $\lambda = \cosh^{-1}(k)$. Other ranges of values for j and k result in altered forms of (4b) as shown in (2d) and (2e) above. In some instances, j and k are complex conjugates yielding the following results:

$$Y_x = \left[\frac{\cos\varphi\sin\varphi\,(x-a-2)\sinh\eta\,(x-a)}{\cosh\eta+\cos\varphi} + \frac{\cosh\eta\sinh\eta\,(x-a-2)\sin\varphi\,(x-a)}{\cosh\eta+\cos\varphi}\right.$$

$$\left.-\sinh\eta\,(x-a-1)\sin\varphi\,(x-a-1)+\sinh\eta\sin\varphi\right]\frac{U\,(x-a)}{4\sinh\eta\sin\varphi\,(\cosh\eta-\cos\varphi)^2}$$

where $\log_e(j+\sqrt{j^2-1})=\eta+i\varphi$.   (4c)

Given $(E^2-2jE+1)^2\,Y_x=U\,(x-a)$, a limiting case of (4a), then, for $j^2<1$,

$$Y_x = \frac{1}{4\sin^2\varphi}\left\{[(\cos\varphi+1)+(x-a)(\cos 2\varphi-\cos\varphi)]\frac{\cos\varphi\,(x-a)-1}{\cos\varphi-1}\right.$$

$$\left.+(x-a)(2\cos\varphi+1)-[(\cos\varphi+2)+(x-a)(\cos 2\varphi+\cos\varphi)]\frac{\sin\varphi\,(x-a)}{\sin\varphi}\right\}U\,(x-a),$$

where $\cos\varphi=j$.

Given $(E^2-2jE+1)(E^2-2kE+1)(E^2-2mE+1)\,Y_x=U\,(x-a)$, then, for j, k and $m>1$,

$$Y_x = \frac{U\,(x-a)}{8\,(j-k)(j-m)}\left[(4j^2-2j-1)\frac{\cosh\varphi\,(x-a)-1}{\cosh\varphi-1}-(4j^2+2j-1)\frac{\sinh\varphi\,(x-a)}{\sinh\varphi}\right]$$

$$+\frac{U\,(x-a)}{8\,(k-j)(k-m)}\left[(4k^2-2k-1)\frac{\cosh\eta\,(x-a)-1}{\cosh\eta-1}\right.$$

$$\left.-(4k^2+2k-1)\frac{\sinh\eta\,(x-a)}{\sinh\eta}\right]$$

$$+\frac{U\,(x-a)}{8\,(m-j)(m-k)}\left[(4m^2-2m-1)\frac{\cosh\sigma\,(x-a)-1}{\cosh\sigma-1}\right.$$

$$\left.-(4m^2+2m-1)\frac{\sinh\sigma\,(x-a)}{\sinh\sigma}\right],$$ (5)

where $\cosh\varphi=j$, $\cosh\eta=k$, and $\cosh\sigma=m$. As above, one can deduce from formulas (2d) and (2e) the necessary changes in form to deal with cases where j, k or $m<+1$.

Given $(E^2-2jE+1)(E^2-2kE+1)(E^2-2mE+1)(E^2-2pE+1)\,Y_x=U\,(x-a)$, then, for j, k, m, and $p>+1$

$$Y_x = \frac{U\,(x-a)}{16\,(j-k)(j-m)(j-p)}\left[(8j^3-4j^2-4j+1)\frac{\cosh\varphi\,(x-a)-1}{\cosh\varphi-1}\right.$$

$$\left.-(8j^3+4j^2-4j-1)\frac{\sinh\varphi\,(x-a)}{\sinh\varphi}\right]$$ (6)

$$+[\,]+[\,]+[\,] \quad\text{(three more similar terms in } k, m, \text{ and } p \text{ respectively)},$$

where $\cosh\varphi=j$ etc.

Given $(E^2-2jE+1)(E^2-2kE+1)(E^2-2mE+1)^2\,Y_x=U\,(x-a)$, a limiting case of (6), then, for j, k and $m>+1$,

$$Y_x = \frac{U(x-a)}{16(j-k)(j-m)^2} \left[(8j^3 - 4j^2 - 4j + 1) \frac{\cosh\varphi(x-a) - 1}{\cosh\varphi - 1} \right.$$

$$\left. - (8j^3 + 4j^2 - 4j - 1) \frac{\sinh\varphi(x-a)}{\sinh\varphi} \right] + [\text{like terms replacing } j \text{ by } k]$$

$$+ \left[\left\{ a_m \left[(m^2 - 1)(j + k - 2m) - \frac{1}{2}(m-j)(m-k) \right] - \frac{3}{2} b_m (m-j)(m-k) \right. \right.$$

$$\left. + a'_m (m-1)(m-j)(m-k)(2+a-x) \right\} \frac{\cosh\sigma(x-a) - 1}{\cosh\sigma - 1} \qquad (6\,a)$$

$$- \left\{ a'_m \left[(m^2 - 1)(j + k - 2m) - \frac{1}{2}(m-j)(m-k) \right] - \frac{3}{2} b'_m (m-j)(m-k) \right.$$

$$\left. + a_m (m+1)(m-j)(m-k)(2+a-x) \right\} \frac{\sinh\sigma(x-a)}{\sinh\sigma}$$

$$\left. - a'_m (m-j)(m-k)(x-a) \right] \frac{U(x-a)}{16(m^2-1)[(m-j)(m-k)]^2} ,$$

where $\quad \cosh\varphi = j; \quad \cosh\sigma = m; \quad a_m = 8m^3 - 4m^2 - 4m + 1;$

$a'_m = 8m^3 + 4m^2 - 4m - 1; \quad b_m = 4m^2 - 2m - 1; \quad \text{and} \quad b'_m = 4m^2 + 2m - 1.$

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[Summary

Closed form solutions were obtained for three curved lattice systems with n elements, a conical lattice with axial symmetry, a circularly arched triangulated truss and a circularly arched Vierendeel truss. The Kronecker delta, δ_x^a, was used as a load shape function throughout so that the resultant working formulæ are complete influence functions and through superposition serve as total solutions. The use of the Kronecker delta as a loading function was introduced and illustrated in detail through solving, with δ_x^v as the inhomogeneous term, the system of difference equations serving as a mathematical model for the polar grid and straight Vierendeel truss. A list of formulas for solving difference equations encountered in lattice analyses is contained in the appendices.

Résumé

L'auteur indique sous forme finie les solutions pour trois systèmes réticules incurvés à n éléments: coupole réticulée conique à symétrie de révolution, treillis incurvé à axe circulaire et cadre Vierendeel incurvé à axe circulaire. Dans ces trois cas, l'auteur a adopté comme fonction de charge le δ_x^a de Kronecker, de sorte que les formules d'application ainsi obtenues constituent des fonctions d'influence complètes, qui permettent d'obtenir des solutions complètes avec intervention de la superposition. L'emploi du δ_x^a de Kronecker, introduit a titre de fonction de charge, est tout d'abord mis en évidence d'une manière détaillée par la résolution, avec δ_x^a à titre de terme inhomogène, du système d'équations aux différences qui sert de modèle mathématique pour la grille polaire et la poutre Vierendeel droite. L'auteur donne ensuite une série de formules pour la résolution des équations aux différences que l'on rencontre dans l'étude des systèmes réticulés.

Zusammenfassung

Für drei gekrümmte Stabsysteme mit n Elementen, ein konischer rotationssymmetrischer Rost, ein kreisförmig gebogener Fachwerkträger und ein kreisförmig gebogener Vierendeelrahmen, werden hier die Lösungen in geschlossener Form angegeben. Überall wurde das Kronecker δ_x^a als Belastungsfunktion verwendet, so daß die daraus erhaltenen Gebrauchsformeln komplette Einflußfunktionen sind, die mit Anwendung der Superposition zu Gesamtlösungen werden. Die Verwendung des Kronecker δ_x^a als Belastungsfunktion wurde eingeführt und im Detail veranschaulicht durch die Lösung, mit δ_x^a als der inhomogene Anteil, des Differenzengleichungssystems, das als mathematisches Modell für den Polarrost und den geraden Vierendeelträger dient. Im Anhang wird eine Formelsammlung für die Lösung von Differenzengleichungen, denen man in der Untersuchung von Stabsystemen begegnet, gegeben.