Elasto-plastic analysis of an interconnected beam systems

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Elasto-Plastic Analysis of an Interconnected Beam System

Calcul élasto-plastique d'une grille de poutres

Elasto-plastische Untersuchung eines Trägerrostes

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Introduction

This paper describes a method for estimating the ultimate load of torsionless interconnected beam systems. As structures of this kind may develop large deflections long before they can be considered to be reduced to a mechanism by the formation of plastic hinges the solution of the problem requires the study of elasto-plastic behaviour.

Stüssi and Kollbrunner [1] pointed out the limitations of the simple plastic theory using the idealized moment-curvature relationship. By varying the relative stiffness of the centre and end spans of a three span continuous beam loaded in the central span they obtained a wide range of collapse load values for the same type of structure. This problem was further investigated by Hodge [2] who expressed the deflection at the load point as a function of ratio of the end span to central span and thus demonstrated quantitatively the effect of deflection on collapse load. Similarly, Hendry [3] tested rectangular portals subjected to central vertical loads and showed that keeping the span constant and varying the height of the frame did not affect the calculated collapse load values but led to greatly increased deflections at the ultimate load end in the elasto-plastic range.

The exact nature of the problem has been very well described by Heyman [4] who discussed plastic behaviour from the viewpoint of the three basic design criteria — I. strength, II. stiffness and III. stability. Simple plastic design methods are concerned only with the strength of the structure and do not attempt to consider collapse in terms of the deflection. The mechanism condition which replaces the requirements of compatibility in elastic theory requires that sufficient plastic hinges must form to turn the structure or part of it into a mechanism of one or more degrees of freedom but does not specify deflection limits. However it tacitly assumes unaltered geometry which is only valid for small deflections.

During recent years considerable interest has been evinced in the analysis of grid frameworks. The method of GUYON [5] and MASSONNET [6] and that of HENDRY and JAEGER [7] offer easy solutions to the elastic analysis of interconnected beam systems. HEYMAN [8, 9] applied the Mechanism solutions developed by SYMONDS and NEAL [10] to the analysis of grids supported on all four sides and subjected to uniform loading. The loads calculated were confirmed by experiment and were within the upper and lower bound limits defined by GREENBERG and PRAGER [11]. HEYMAN also indicated an iterative procedure for improving the estimate of the collapse load. HAYTHORNTHWAITE [12] studied the elasto-plastic behaviour of certain simple grids. However, this work was limited to three longitudinal grids with loading symmetrical about the longitudinal axis. HAYTHORNTHWAITE suggested a modified mechanism method for computing the ultimate load wherein any hinge which depended for its formation on the torsional rigidity of a supporting member was replaced by a mechanical hinge. The work of HEYMAN and HAYTHORNTHWAITE was in a way limited by the difficulties of elastic analysis. The chief advantage of the method of HENDRY and JAEGER is the presentation of results in a parametric form. The distribution of moments and deflections in the various members is determined by distribution factors which are expressed as functions of the flexural and torsional parameters. Apart from the effective starting point if provides for the elasto-plastic analysis the HENDRY-JAEGER method permits the consideration of the concept of "Moment ratios" which is of importance in elasto-plastic behaviour.

The method of elasto-plastic analysis developed in this paper will be as follows: the elastic solution is first worked out and the cross-section where the moment reaches the yield value for a load P_y is determined. A plastic hinge is assumed to have formed at the section and any subsequent increase in loading ΔP_y is assumed to cause the hinge to undergo rotation while the bending moment remains constant at the value of the fully plastic moment at the section. Elsewhere the structure will behave elastically. Thus the subsequent increments of bending moment and deflection will be the same as those which would be caused by loads ΔP_y applied to the frame if it were behaving elastically but with a mechanical hinge at the section where the plastic hinge had formed.

The general approach in the analysis will be to express the incremental loading as a function of an arbitrary displacement at the load point. Since deflection is the limiting factor in the determination of the ultimate load for a grid the moments and deflections are expressed as functions of the incremental deflections Δ and the flexural stiffness parameter α. Each new hinge reduces the degree of indeterminacy by one and transforms the grid into a new structure which will be termed the "Reduced Grid". For a given value of the flexural parameter α the solutions of the so called "Reduced Grids" can be combined to establish the elasto-plastic behaviour under a certain loading system.

The concept of the "plastic hinge" as used in this analysis has caused some controversy. However, an assessment of the extensive analytical and experimental research work on steel structures carried out both in Europe and the U.S.A. indicates that the assumptions of the simple plastic hinge are sufficiently accurate for practical purposes in the present context. The significance of the assumptions made in the simple plastic theory has sometimes been misconstrued. KUZMANOVIĆ [13] has criticised the phenomenon of redistribution of moments in an indeterminate structure and suggested that the tests at Cambridge by Baker were not convincing in showing moment redistribution because the differences in the moment values were small. However, a test by YANG, BEEDLE and JOHNSTON [14] on a fixed-ended beam with third point loading showed clear evidence of redistribution of moments although the centre moment is only one half of the moment at the fixed ends. Of course a plastic design which takes into account strain hardening would be more desirable but the complexity of the problem would be considerable. It was Aristotle who pointed out the merit of resting satisfied with that degree of precision which the nature of the subject admits and not seeking exactness where only an approximate solution is possible.

Analysis of a Four Girder Grid

The method of analysis will be illustrated by considering a No-Torsion grid with four longitudinals and three transversals loaded as shown in Fig. 1. The possible modes of elasto-plastic behaviour are shown in Fig. 2. The particular "mode" of elasto-plastic behaviour is completely determined in terms of the flexural parameter α and there is no need for trial and error. Since the method of analysis is the same for different modes it is sufficient to analyse the load-deflection behaviour of one mode only.

The analysis will now be described for a particular mode but with alternative hinge positions in Stage I.

Stage I

$$\text{Deflection } Y'_{2B} = \Delta_1. \qquad \text{Load at } 2B = P_1.$$

Two methods of analysis can be used for this stage

a) Hendry-Jaeger Method [15]. By replacing the transverse members of a grid by a uniformly spread medium and applying harmonic analysis to the loading distribution coefficients for the various longitudinals are determined.

The moments and deflections in the various members of the frame are found using these distribution coefficients which are functions of a flexural stiffness parameter. This parameter

$$\alpha = \frac{12}{\pi^4}\left(\frac{l}{h}\right)^3 \frac{n\,E\,I_T}{E\,I_L}$$

emerges from the harmonic analysis solution. As the method has been described in the reference cited it will not be necessary to discuss it in detail here.

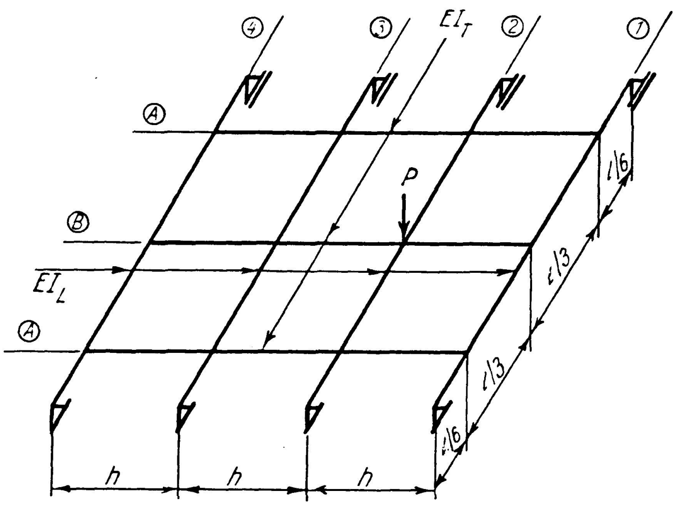

Fig. 1.

b) Slope-deflection Method. The force system is shown in Fig. 3. The compatibility conditions for the deflections of transversals and longitudinals are easily expressed in terms of the moment-deflection relationships of the transversals derived by Covington [16] from the well known Slope-Deflection equations.

The transverse moments M'_{2B_T}, M'_{3B_T}, M'_{2A_T}, M'_{3A_T} are treated as unknowns. Considering sagging moments to be positive the moment deflection relationships for the transversals may be expressed as follows (Fig. 4a).

$$M'_{2B_T} = \frac{2\,E\,I_T}{5\,h^2}(9\,Y'_{2B} - 6\,Y'_{3B} + Y'_{4B} - 4\,Y'_{1B}), \tag{1a}$$

$$M'_{3B_T} = \frac{2\,E\,I_T}{5\,h^2}(9\,Y'_{3B} - 6\,Y'_{2B} + Y'_{1B} - 4\,Y'_{4B}), \tag{1b}$$

$$M'_{2A_T} = \frac{2\,E\,I_T}{5\,h^2}(9\,Y'_{2A} - 6\,Y'_{3A} + Y'_{4A} - 4\,Y'_{1A}), \tag{1c}$$

$$M'_{3A_T} = \frac{2\,E\,I_T}{5\,h^2}(9\,Y'_{3A} - 6\,Y'_{2A} + Y'_{1A} - 4\,Y'_{4A}). \tag{1d}$$

For longitudinals (1), (3) and (4) the deflections at transversal positions (A) and (B) are as follows:

$$Y_B' = \frac{l^3}{E I_L}\left(\frac{1}{48} W_B + \frac{13}{648} W_A\right), \tag{2a}$$

$$Y_A' = \frac{l^3}{E I_L}\left(\frac{13}{1296} W_B + \frac{7}{648} W_A\right). \tag{2b}$$

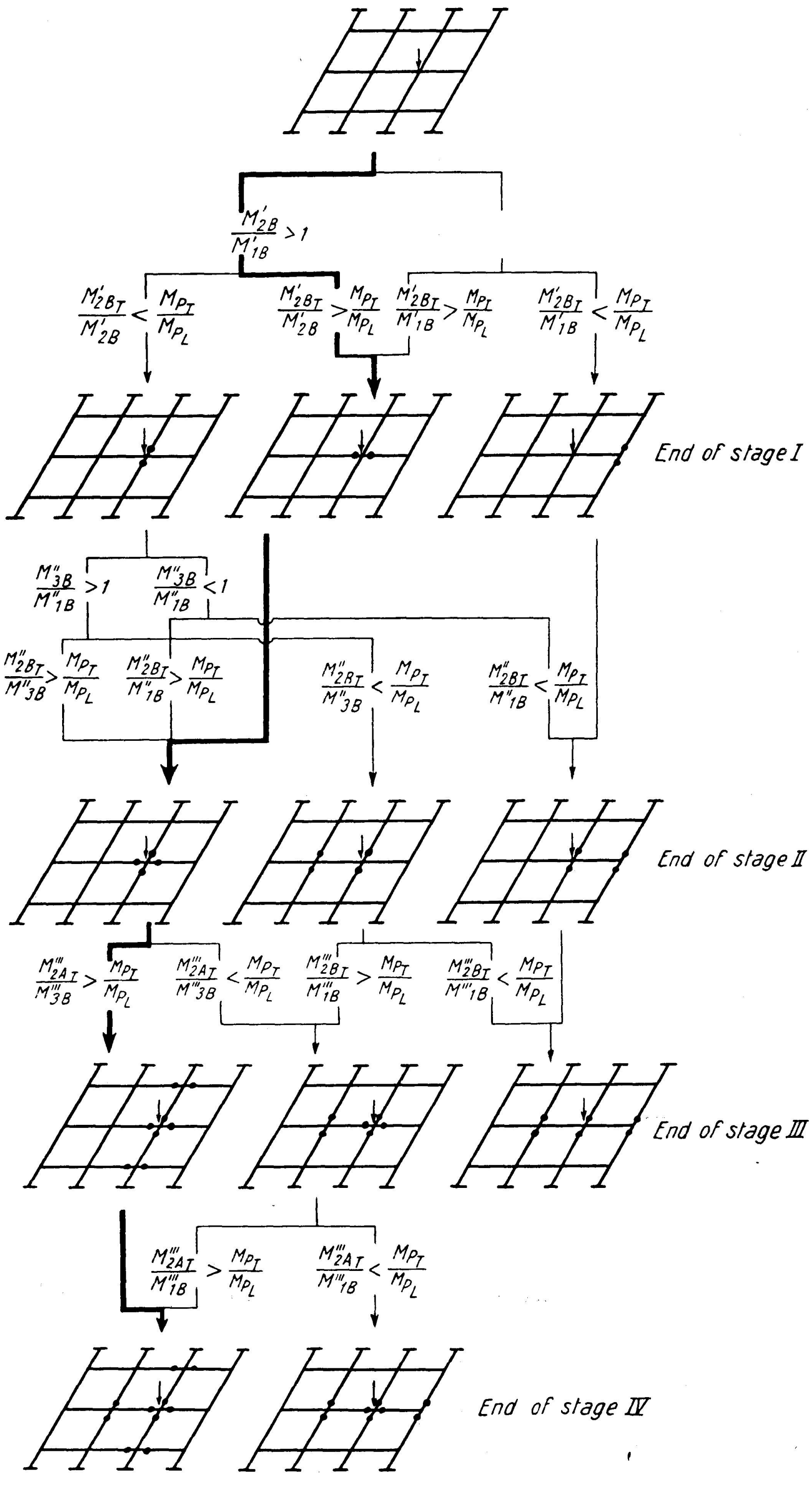

Fig. 2. Modes of Elasto-Plastic Behaviour.

Where W_B and W_A refer to the loads at sixth points A and centres B of any particular longitudinal.

For longitudinal (2) the deflection Y'_{2A} is expressed as a function of $Y'_{2B} = \Delta_1$ by means of the simple Moment-area relationships.

$$Y'_{2A} = \frac{13}{27}\Delta_1 - \frac{1}{486}\frac{l^3}{E I_L}\left(2\frac{M'_{2A_T}}{h} - \frac{M'_{3A_T}}{h}\right). \tag{3}$$

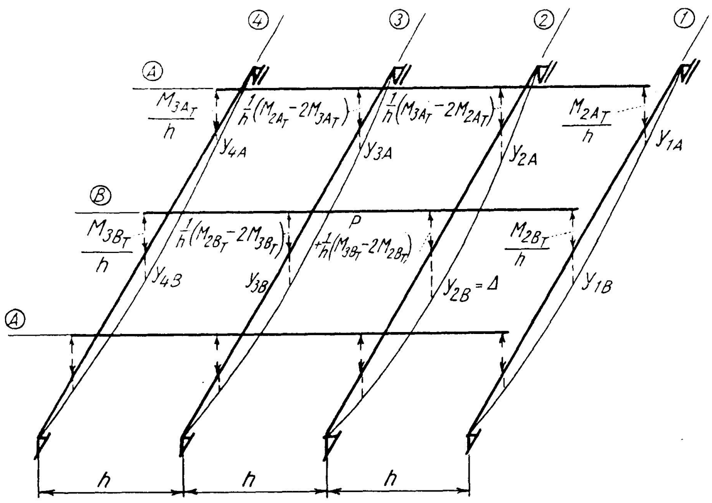

Fig. 3. Force System.

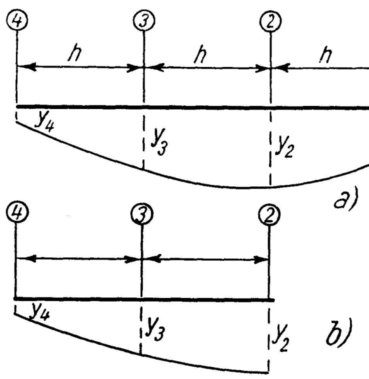

Fig. 4.

Sign Convention:

Sagging Moments
Downward Deflections } Positive.
Downward Forces

Substituting the values of Y'_{2B}, Y'_{3B}, Y'_{2A} and Y'_{3A} in Eqs. (1) and introducing α which is the same as that used by HENDRY and JAEGER, the following equations are obtained:

$$\begin{bmatrix} \left(-270 - \frac{116{,}640}{\pi^4}\frac{1}{\alpha}\right) & +351 & -260 & +338 \\ +270 & \left(-594 - \frac{116{,}640}{\pi^4}\frac{1}{\alpha}\right) & +260 & -572 \\ -130 & +169 & \left(-188 - \frac{116{,}640}{\pi^4}\frac{1}{\alpha}\right) & +206 \\ +130 & -286 & +172 & \left(-324 - \frac{116{,}640}{\pi^4}\frac{1}{\alpha}\right) \end{bmatrix} \cdot$$

$$\cdot \begin{bmatrix} M'_{2B_T} \\ M'_{3B_T} \\ M'_{2A_T} \\ M'_{3A_T} \end{bmatrix} = \frac{E I_L h \Delta_1}{l^3} \begin{bmatrix} -11{,}664 \\ +\ 7{,}776 \\ -\ 5{,}616 \\ -\ 3{,}744 \end{bmatrix}. \tag{4}$$

Solution of the above equations gives values of M'_{2B_T}, M'_{3B_T}, M'_{2A_T} and M'_{3A_T} as functions of Δ_1 and α. The longitudinal moments and deflections are easily obtained from the force system and are tabulated in the Appendix.

The position of the first hinge is now determined as follows:

a) If $\dfrac{M'_{2B}}{M'_{1B}} > 1$ then the hinge occurs at $2B$,

and if $\dfrac{M'_{2B_T}}{M'_{2B}} > \dfrac{M\,p_T}{M\,p_L}$ hinge in transversal (B) at $2B$,

$\dfrac{M'_{2B_T}}{M'_{2B}} = \dfrac{M\,p_T}{M\,p_L}$ hinges in both longitudinal (2) and transversal (B) at $2B$,

$\dfrac{M'_{2B_T}}{M'_{2B}} < \dfrac{M\,p_T}{M\,p_L}$ hinge in longitudinal at $2B$.

b) If $\dfrac{M'_{2B}}{M'_{1B}} < 1$ then the first hinge occurs at $1B$ and the following condition must be considered

$$\frac{M'_{2B_T}}{M'_{1B}} \gtrless \frac{M\,p_T}{M\,p_L}.$$

c) If $\dfrac{M'_{2B}}{M'_{1B}} = 1$ then hinges occur simultaneously at $1B$ and $2B$.

When the position of the first hinge is located the value of the deflection is formed and therefore all the moment and deflection values are easily calculated.

Moments evaluated from the HENDRY-JAEGER method can be used in the investigation of "Moment Inequalities". The HENDRY-JAEGER solutions express moments as functions of distribution coefficients which are themselves functions of α. The distribution coefficients have been presented in the form of tables and curves which enables easy application to design problems.

Stage II

$$\text{Deflection } y''_{2B} = \delta_2, \qquad \text{Load} = P_2 - P_1.$$

The following alternatives have to be considered:

a) hinge in longitudinal at $2B$,

b) hinge in transversal at $2B$.

Case a. The equations are formed by the same method as that in Stage I. The only difference is in the equation of the elastic line for longitudinal (2) for now the longitudinal moment $m''_{2A} = 0$.

$$y''_{2A} = \frac{1}{3}\delta_2 - \frac{1}{486}\frac{l^3}{E I_L}\left(2\frac{m''_{2A_T}}{h} - \frac{m''_{3A_T}}{h}\right), \tag{5}$$

$$\begin{bmatrix} \left(-270-\frac{116{,}640}{\pi^4}\frac{1}{\alpha}\right) & +351 & -260 & +338 \\ +270 & \left(-594-\frac{116{,}640}{\pi^4}\frac{1}{\alpha}\right) & +260 & -572 \\ -130 & +169 & \left(-188-\frac{116{,}640}{\pi^4}\frac{1}{\alpha}\right) & +206 \\ +130 & -286 & +172 & \left(-324-\frac{116{,}640}{\pi^4}\frac{1}{\alpha}\right) \end{bmatrix} \cdot \begin{bmatrix} m''_{2B_T} \\ m''_{3B_T} \\ m''_{2A_T} \\ m''_{3A_T} \end{bmatrix} = \frac{E I_L h \delta_2}{l^3} \begin{bmatrix} -11{,}664 \\ +\ 7{,}776 \\ -\ 3{,}888 \\ +\ 2{,}592 \end{bmatrix}. \tag{6}$$

The solutions are tabulated in the Appendix as before.

The following "Moment Inequalities" have to be examined

1. If $$\frac{M''_{3B}}{M''_{1B}} > 1$$

consider $$\frac{M''_{2B_T}}{M''_{3B}} \gtrless \frac{M p_T}{M p_L}.$$

2. If $$\frac{M''_{3B}}{M''_{1B}} < 1$$

consider $$\frac{M''_{2B_T}}{M''_{1B}} \gtrless \frac{M p_T}{M p_L}.$$

The incremental moment values m''_{2B_T}, m''_{1B} and m''_{3B} are functions of the unknown incremental deflection δ_2 and the above inequalities are of the form

$$\frac{p+q\,\delta_2}{r+s\,\delta_2} \gtrless k,$$

which can be expressed as $$\delta_2 \gtrless \frac{k r - p}{q - k s},$$

where p, q, r, s and k are known values.

Case b. The transverse moment $m''_{2B_T} = 0$ and the number of unknowns is therefore reduced by one.

The deflection y''_{2A} is of the same form as that in Stage I and is as follows:

$$y''_{2A} = \frac{13}{27}\delta_2 - \frac{1}{486}\frac{l^3}{E I_L}\left(2\frac{m''_{2A_T}}{h} - \frac{m''_{3A_T}}{h}\right). \tag{7}$$

The moment-deflection relationship of transversal B is expressed as (Fig. 4b)

$$m''_{3B_T} = \frac{3 E I_T}{2 h^2}\left[2\, y''_{3B} - (y''_{4B} + y''_{2B})\right]. \tag{8}$$

The moment-deflection relationships for transversal A are similar to those in Stage I, viz. (1c), (1d).

The equations are as follows:

$$\begin{bmatrix} \left(-135-\frac{31{,}104}{\pi^4}\frac{1}{\alpha}\right) & +52 & -130 \\ +169 & \left(-188-\frac{116{,}640}{\pi^4}\frac{1}{\alpha}\right) & +206 \\ -286 & +172 & \left(-324-\frac{116{,}640}{\pi^4}\frac{1}{\alpha}\right) \end{bmatrix} \cdot \begin{bmatrix} m'''_{3B_T} \\ m'''_{2A_T} \\ m'''_{3A_T} \end{bmatrix} = \frac{E I_L \delta_3 h}{l^3} \begin{bmatrix} +1296 \\ -5616 \\ +3744 \end{bmatrix}. \quad (9)$$

No moment inequalities need to be considered at this stage since it is obvious that the next hinge occurs at 2 B in the longitudinal.

Stage III

$$\text{Deflection } y'''_{2B} = \delta_3, \qquad \text{Load} = P_3 - P_2.$$

The moment-deflection relationships for transversals are similar in form to those in Stage IIb.

The equations are as follows:

$$\begin{bmatrix} \left(-135-\frac{31{,}104}{\pi^4}\frac{1}{\alpha}\right) & +52 & -130 \\ +169 & \left(-188-\frac{116{,}640}{\pi^4}\frac{1}{\alpha}\right) & +206 \\ -286 & +172 & \left(-324-\frac{116{,}640}{\pi^4}\frac{1}{\alpha}\right) \end{bmatrix} \cdot \begin{bmatrix} m'''_{3B_T} \\ m'''_{2A_T} \\ m'''_{3A_T} \end{bmatrix} = \frac{E I_L \delta_3 h}{l^3} \begin{bmatrix} +1296 \\ -3888 \\ +2592 \end{bmatrix}. \quad (10)$$

The moment inequality to be considered is

$$\frac{M'''_{2A_T}}{M'''_{3B}} \gtrless \frac{M p_T}{M p_L}.$$

Stage IV

$$\text{Deflection } y''''_{2B} = \delta_4, \qquad \text{Load} = P_4 - P_3,$$

$$m''''_{2B_T} = m''''_{2A_T} = 0, \qquad m''''_{2B} = 0.$$

The moment-deflection equations are

$$m''''_{3B_T} = \frac{3\,E\,I_T}{2\,h^2}[2\,y''''_{3B} - (y''''_{4B} + y''''_{3B})], \tag{11}$$

$$m''''_{3A_T} = \frac{3\,E\,I_T}{2\,h^2}[2\,y''''_{3A} - (y''''_{4A} + y''''_{2A})]. \tag{12}$$

The equations are

$$\begin{bmatrix} \left(-135 - \frac{31{,}104}{\pi^4}\frac{1}{\alpha}\right) & -130 \\ -65 & \left(-\frac{218}{3} - \frac{31{,}104}{\pi^4}\frac{1}{\alpha}\right) \end{bmatrix} \begin{bmatrix} m''''_{3B_T} \\ m''''_{3A_T} \end{bmatrix} = \frac{E\,I_L\,\delta_4\,h}{l^3}\begin{bmatrix} +1296 \\ +\ 432 \end{bmatrix}. \tag{13}$$

The collapse load is $P_u = P_4$ and the deflection at the load point is

$$\Delta_4 = \Delta_1 + \delta_2 + \delta_3 + \delta_4.$$

Experimental Confirmation

The analytical solutions were verified by application to a Model grid which was tested to collapse (Fig. 5). The properties of the grid tested were as follows:

$$1 = 36'' \qquad h = 12''$$

Longitudinals: H sections $2'' \times 1'' \times 2.5$ lbs.

From control tests $E\,I_L = 121.25 \times 10^5$ lb. in.^{2}

$M\,p_L = 23{,}300$ in. lbs.

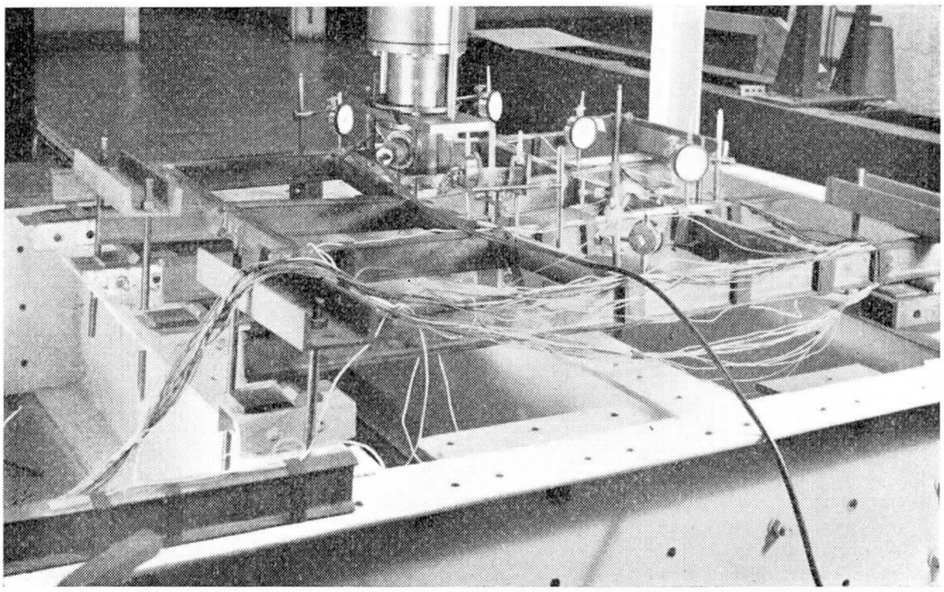

Fig. 5. Test Set-up.

Transversals: H sections $1\frac{1}{2}'' \times \frac{3}{4}'' \times 1.5$ lbs.

From control tests $E I_T = 53 \times 10^5$ lb. in.^{2}

$$M p_T = 7680 \text{ in. lbs.}$$

$$\alpha = 3.26$$

The grid was instrumented with both deflection and resistance strain gauges. The strain gauge readings confirmed the sequence of hinges obtained from the analytical procedure. However it was difficult to define the exact limits of the successive stages from the experimental data. The load deflection relationship for the loaded longitudinal is given in Fig. 6 and those for the unloaded longitudinals are given in Fig. 7. The values obtained from the analytical solution are shown in dotted lines.

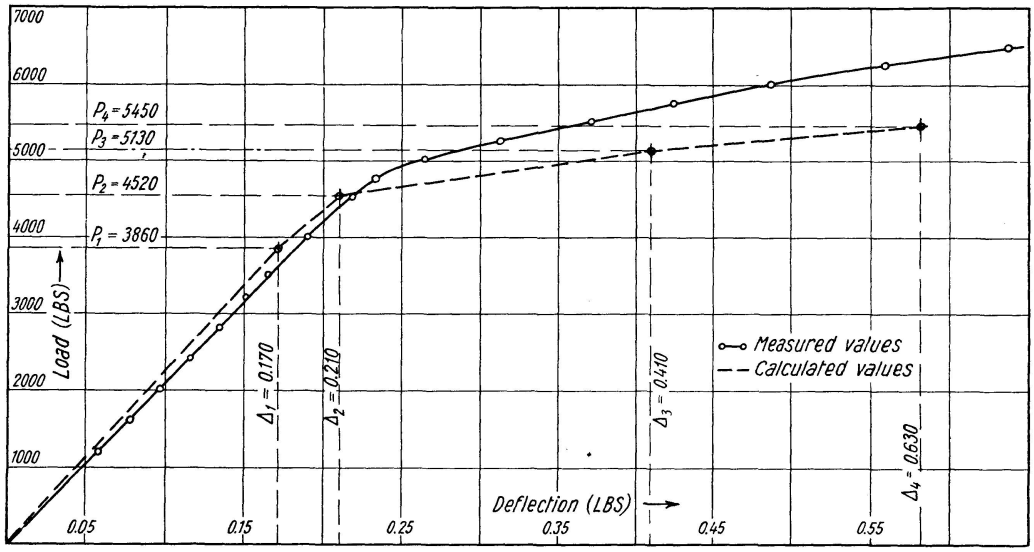

Fig. 6. Load-Deflection Curves at Mid-span of Longitudinal (2).

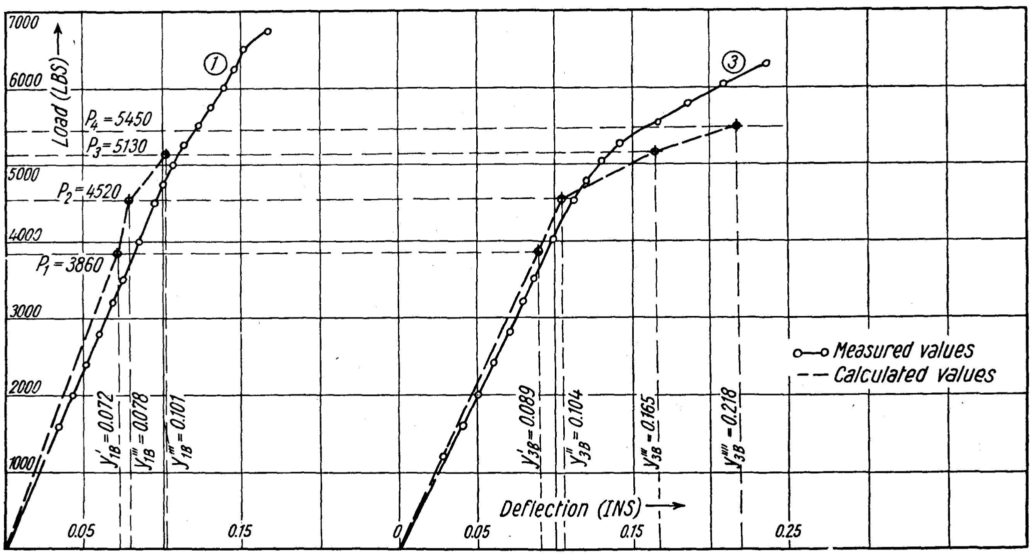

Fig. 7. Load-Deflection Curves at Mid-spans of Longitudinals (1) and (3).

Discussion

It will be observed from the foregoing that there is reasonable agreement between the experimental results and the values obtained from the analytical method outlined in the paper. A mechanism solution of the grid gives the value of the collapse load as

$$P_u = \frac{6\,M\,p_L}{l} + \frac{5}{2}\frac{M\,p_T}{h}, \qquad (14)$$
$$= 5500\,\text{lbs.}$$

This value corresponds to the value of $P_4 = 5450$ lbs. obtained from elasto-plastic analysis. A deflection value of 0.630 in., which is indeed large for a span of 36″, is necessary for this load to be reached. If a permissible central displacement is specified the corresponding load can be calculated and it will not be necessary to calculate all the stages of the elasto-plastic analysis.

Space does not permit discussion of other analytical and experimental work on different types of grids with various loading conditions including multiple loads. When the degree of indeterminacy becomes greater suitable approximations can be made to simplify the analysis. For example, a member with a plastic hinge can be neglected in the succeeding stages of the elasto-plastic behaviour of the structure. The degree of accuracy of the analytical values thus obtained can be improved by an iteration procedure similar to that described by REDDY and HENDRY [17]. The stage by stage method of elasto-plastic analysis can be easily programmed for an electronic computer by a procedure similar to that used by LIGHTFOOT and SAWKO [18]

Notation

$E\,I_L$, $E\,I_T$	= Flexural rigidity of longitudinal and transversal respectively.
h	= Spacing of longitudinals.
l	= Span of grid.
$M'_{ij}, M''_{ij}, M'''_{ij}, M''''_{ij}$	= Moments in longitudinal i at its intersection with transversal j when a number of hinges corresponding to the number of primes are formed.
$m''_{ij}, m'''_{ij}, m''''_{ij}$	= Incremental moments $(M''_{ij} - M'_{ij})$, $(M'''_{ij} - M''_{ij})$, $(M''''_{ij} - M'''_{ij})$ respectively.
$M'_{ij_T}, M''_{ij_T}, M'''_{ij_T}, M''''_{ij_T}$, $m''_{ij_T}, m'''_{ij_T}, m''''_{ij_T}$	= Moments and Incremental moments in transversal j at intersection $i\,j$.
$M\,p_L$	= Full plastic moment of longitudinal.
$M\,p_T$	= Full plastic moment of transversal.
n	= Number of transversals.

P_1, P_2, P_3, P_4	= Concentrated load at $2B$ for first, second, third and fourth hinges respectively.
P_u	= Ultimate load.
$Y'_{ij}, Y''_{ij}, Y'''_{ij}, Y''''_{ij}$	= Deflections at ij when a number of hinges corresponding to the number of primes are formed.
$y''_{ij}, y'''_{ij}, y''''_{ij}$	= Incremental deflections $(Y''_{ij} - Y'_{ij})$, $(Y'''_{ij} - Y''_{ij})$, $(Y''''_{ij} - Y'''_{ij})$.
α	= Flexural parameter $= \frac{12}{\pi^4}\left(\frac{l}{h}\right)^3 \frac{n E I_T}{E I_L}$.
$\Delta_1, \Delta_2, \Delta_3, \Delta_4$	= Displacements at load point at the formation of first, second, third and fourth hinges.
$\delta_2, \delta_3, \delta_4$	= Incremental deflections at load point $(\Delta_2 - \Delta_1)$, $(\Delta_3 - \Delta_2)$, $(\Delta_4 - \Delta_3)$ respectively.

Appendix

Stage I

$$\text{Load: } P_1 = \frac{E I_L \Delta_1}{l^3 D_1}(0.1776\,\alpha^4 + 34.50\,\alpha^3 + 700.4\,\alpha^2 + 3023\,\alpha + 1622)$$

$$\text{where } D_1 = 0.001110\,\alpha^4 + 0.2281\,\alpha^3 + 5.849\,\alpha^2 + 38.83\,\alpha + 33.79$$

$$\text{Longitudinal Moments: } M'_{ij} = \frac{E I_L \Delta_1}{l^2 D_1}(a\,\alpha^4 + b\,\alpha^3 + c\,\alpha^2 + d\,\alpha + e)$$

Table I-A

M'_{ij}	Coefficients				
	a	b	c	d	e
M'_{1A}	–0.005928	1.347	26.68	53.84	
M'_{1B}	–0.01775	3.254	55.92	108.7	
M'_{2A}	0.004450	0.6640	9.330	108.4	135.2
M'_{2B}	0.01332	2.737	70.19	466.0	405.5
M'_{3A}	–0.008890	0.3812	18.02	125.6	
M'_{3B}	–0.07092	2.016	41.99	183.2	
M'_{4A}	0.001481	0.4828	4.330	–35.90	
M'_{4B}	0.004437	0.6186	6.962	–72.48	

$$\text{Transverse Moments: } M'_{ij_T} = \frac{E I_L \Delta_1 h}{l^3 D_1}(a\,\alpha^4 + b\,\alpha^3 + c\,\alpha^2 + d\,\alpha)$$

Table I-B

M'_{ij_T}	Coefficients			
	a	b	c	d
M'_{2B_T}	0.07107	11.44	175.5	329.2
M'_{3B_T}	0.01774	0.8145	15.80	–219.4
M'_{2A_T}	–0.000031	2.362	72.34	158.5
M'_{3A_T}	0.000020	2.489	18.08	–105.7

Deflections: $Y'_{ij} = \frac{\Delta_1}{D_1}(a\,\alpha^4 + b\,\alpha^3 + c\,\alpha^2 + d\,\alpha + e)$

Table I-C

Y'_{ij}	Coefficients				
	a	b	c	d	e
Y'_{1A}	0.0007127	0.1402	2.541	5.014	
Y'_{1B}	0.001475	0.2857	5.107	10.04	
Y'_{2A}	0.0005342	0.1052	2.556	17.83	16.27
Y'_{2B}	0.001110	0.2281	5.849	38.83	33.79
Y'_{3A}	0.0003563	0.07010	1.834	11.70	
Y'_{3B}	0.0007403	0.1519	3.724	23.42	
Y'_{4A}	0.0001777	0.03506	0.3537	–3.342	
Y'_{4B}	0.0003700	0.06690	0.6918	–6.692	

Stage II, Case a)

Load: $P_2 - P_1 = \frac{P\,I_L\,\delta_2}{l^3\,D_1}(0.1435\,\alpha^4 + 25.77\,\alpha^3 + 413.8\,\alpha^2 + 1073\,\alpha)$

Longitudinal Moments: $m''_{ij} = \frac{E\,I_L\,\delta_2}{l^2\,D_1}(a\,\alpha^4 + b\,\alpha^3 + c\,\alpha^2 + d\,\alpha)$

Table II-A

m''_{ij}	Coefficients			
	a	b	c	d
m''_{1A}	0.002727	0.9774	21.19	45.71
m''_{1B}	0.02051	3.428	53.73	100.6
m''_{2A}	0.007189	0.8325	–2.162	–32.51
m''_{2B}	0	0	0	0
m''_{3A}	0.001356	–0.3027	9.702	110.7
m''_{3B}	0.01026	2.601	45.68	246.7
m''_{4A}	0.000686	0.6401	5.747	–30.48
m''_{4B}	0.005122	0.4133	4.026	–67.05

Transverse Moments: $m''_{ij_T} = \frac{E\,I_L\,\delta_2\,h}{l^3\,D_1}(a\,\alpha^4 + b\,\alpha^3 + c\,\alpha^2 + d\,\alpha)$

Table II-B

m''_{ij_T}	Coefficients			
	a	6	c	d
m''_{2B_T}	0.1066	14.70	195.2	329.2
m''_{3B_T}	0.02661	–1.361	–10.32	–219.4
m''_{2A_T}	–0.03695	–1.486	29.55	109.7
m''_{3A_T}	–0.009191	4.521	39.64	–73.16

Deflections: $y''_{ij} = \frac{\delta_2}{D_1}(a\,\alpha^4 + b\,\alpha^3 + c\,\alpha^2 + d\;\alpha + e)$

Table II-C

y''_{ij}	Coefficients				
	a	b	c	d	e
y''_{1A}	0.000670	0.1314	2.278	4.487	
y''_{1B}	0.001480	0.2765	4.660	9.058	
y''_{2A}	0.000503	0.09144	1.910	12.34	11.26
y''_{2B}	0.001110	0.2281	5.849	38.83	33.79
y''_{3A}	0.000335	0.06105	1.628	35.36	
y''_{3B}	0.000740	0.1518	3.500	67.36	
y''_{4A}	0.000168	0.03518	0.3246	–2.992	
y''_{4B}	0.000370	0.06235	0.5802	–6.039	

Stage II, Case b)

Load: $P_3 - P_2 = \frac{E\,I_L\,\delta_3}{l^3\,D_2}(4.992\,\alpha^3 + 294.2\,\alpha^2 + 2436\,\alpha + 2198)$

where $D_2 = 0.04451\,\alpha^3 + 3.586\,\alpha^2 + 38.93\,\alpha + 45.78$

Longitudinal Moments: $m''_{ij} = \frac{E\,I_L\,\delta_2}{l^2\,D_2}(a\,\alpha^3 + b\,\alpha^2 + c\,\alpha + d)$

Table II-D

m''_{ij}	Coefficients			
	a	b	c	d
m''_{1A}	0.3395	16.34	35.79	
m''_{1B}		$= m''_{1A}$		
m''_{2A}	–0.08809	–8.305	92.11	183.1
m''_{2B}	0.5341	43.03	467.2	549.4
m''_{3A}	–0.009982	16.63	114.5	
m''_{3B}	0.4093	12.03	176.4	
m''_{4A}	0.1747	–0.1456	–39.34	
m''_{4B}	–0.03493	2.153	–70.31	

Transverse Moments: $m''_{ji_T} = \frac{E\,I_L\,\delta_2\,h}{l^3\,D_2}(a\,\alpha^3 + b\,\alpha^2 + c\,\alpha)$

Table II-E

m''_{ij_T}	Coefficients		
	a	b	c
m''_{2B_T}	0	0	0
m''_{3B_T}	–1.258	13.79	–185.8
m''_{2A_T}	2.037	98.03	214.7
m''_{3A_T}	1.677	–7.771	–143.2

Deflections: $y''_{ij} = \frac{\delta_2}{D_2}(a\,\alpha^3 + b\,\alpha^2 + c\,\alpha + d)$

Table II-F

y''_{ij}	Coefficients			
	a	b	c	d
y''_{1A}	0.02200	1.059	2.320	
y''_{1B}	0.0409	1.967	4.308	
y''_{2A}	0.0165	1.307	17.57	22.04
y''_{2B}	0.04451	3.586	38.93	45.78
y''_{3A}	0.0110	1.504	9.140	
y''_{3B}	0.0260	1.704	17.79	
y''_{4A}	0.0055	0.05511	−3.410	
y''_{4B}	0.007444	0.1315	−6.743	

Stage III

Load: $P_3 - P_2 = \frac{E\,I_L\,\delta_3}{l^3\,D_2}(2.558\,\alpha^3 + 89.11\,\alpha^2 + 450.1\,\alpha)$

Longitudinal Moments: $m'''_{ij} = \frac{E\,I_L\,\delta_3}{l^2\,D_2}(a\,\alpha^2 + b\,\alpha + c)$

Table III-A

m'''_{ij}	Coefficients		
	a	b	c
m'''_{1A}	0.2627	9.966	24.78
m'''_{1B}		$= m'''_{1A}$	
m'''_{2A}	−0.1118	−12.37	−44.05
m'''_{2B}	0	0	0
m'''_{3A}	−0.1382	9.691	88.78
m'''_{3B}	0.4908	14.66	150.7
m'''_{4A}	0.2005	0.1379	−32.00
m'''_{4B}	−0.1140	−2.345	−62.97

Transverse Moments: $m'''_{ij_T} = \frac{E\,I_L\,\delta_3\,h}{l^3\,D_2}(a\,\alpha^2 + b\,\alpha + c)$

Table III-B

m'''_{ij_T}	Coefficients		
	a	b	c
m'''_{2B_T}	0	0	0
m'''_{3B_T}	−1.887	−14.90	−185.8
m'''_{2A_T}	1.576	59.80	148.7
m'''_{3A_T}	2.146	8.277	−99.11

$$\text{Deflections: } y'''_{ij} = \frac{\delta_3}{D_2}(a\,\alpha^3 + b\,\alpha^2 + c\,\alpha + d)$$

Table III-C

y'''_{ij}	Coefficients			
	a	b	c	d
y'''_{1A}	0.0170	0.6459	1.606	
y'''_{1B}	0.03162	1.200	2.982	
y'''_{2A}	0.01277	0.9662	12.16	15.26
y'''_{2B}	0.04451	3.5860	38.93	45.78
y'''_{3A}	0.008228	0.7660	7.474	
y'''_{3B}	0.02413	1.488	14.70	
y'''_{4A}	0.004256	–0.0600	–2.934	
y'''_{4B}	0.003747	–0.1443	–5.859	

Stage IV

$$\text{Load: } P_4 - P_3 = \frac{E\,I_L\,\delta_4}{l^3\,D_3}(2.074\,\alpha^2 + 50.58\,\alpha)$$

$$\text{where } D_3 = 0.1360\,\alpha^2 + 6.631\,\alpha + 10.20\,\alpha$$

$$\text{Longitudinal Moments: } m''''_{ij} = \frac{E\,I_L\,\delta_4}{l^3\,D_3}(a\,\alpha^2 + b\,\alpha)$$

Table IV-A

m''''_{ij}	Coefficients	
	a	b
m''''_{1A}	0	0
m''''_{1B}	0	0
m''''_{2A}	0.2880	–1.533
m''''_{2B}	0	0
m''''_{3A}	–0.2306	11.49
m''''_{3B}	1.036	39.08
m''''_{4A}	$= -\frac{1}{2}\,m''''_{3A}$	
m''''_{4B}	$= -\frac{1}{2}\,m''''_{3B}$	

$$\text{Transverse Moments: } m''''_{ij_T} = \frac{E\,I_L\,\delta_4\,h}{l^3\,D_3}(a\,\alpha^2 + b\,\alpha)$$

Table IV-B

m''''_{ij_T}	Coefficients	
	a	b
m''''_{2B_T}	0	0
m''''_{3B_T}	–3.802	–41.38
m''''_{2A_T}	0	0
m''''_{3A_T}	2.592	–13.79

$$\text{Deflections: } y_{ij}'''' = \frac{\delta_4}{D_3}(a\alpha^2 + b\alpha + c)$$

Table IV-C

y_{ij}''''	Coefficients		
	a	b	c
y_{1A}''''	0	0	0
y_{1B}''''	0	0	0
y_{2A}''''	0.0506	2.182	3.399
y_{2B}''''	0.1360	6.631	10.20
y_{3A}''''	0.02027	1.128	
y_{3B}''''	0.0544	2.278	
y_{4A}''''		$= -\frac{1}{2}\, y_{3A}''''$	
y_{4B}''''		$= -\frac{1}{2}\, y_{3B}''''$	

Bibliography

1. Stüssi, F., Kollbrunner, C. F., "Beitrag zum Traglast-Verfahren". Bautechnik, Vol. 13, 264 (1935).
2. Hodge, P. G. Jnr., "Plastic Analysis of Structures". McGraw Hill Book Co., New York 89—90 (1959).
3. Hendry, A. W., "An Investigation of the Strength of Certain Welded Portal Frames in relation to the Plastic Method of Design". Struct. Engr., 28, 311 (1950).
4. Heyman, J., "Limit Design of Framed Structures". Engineering, Vol. 187, No. 4851 (1959).
5. Guyon, Y., "Calcul des ponts larges à poutres multiples solidarisées par des entretoises". Ann. Ponts et Chaussees, 553—612 (1946).
6. Massonnet, C., "Méthode de calcul des ponts à poutres multiples tenant compte de leur résistance à la torsion". Publ. of Int. Assn. for Bridge and Structural Engineering. Vol. X, 147—182 (1950).
7. Hendry, A. W., Jaeger, L. G., "The Load Distribution in Interconnected Bridge Girders with Special Reference to Continuous Beams". Publ. of Int. Assn. for Bridge and Structural Engineering, Vol. XV., 95—116 (1955).
8. Heyman, J., "The Limit Design of a Transversely Loaded Square Grid". Journ. Appl. Mechanics, 19, 153—158 (1952).
9. Heyman, J., "The Plastic Design of Grillages". Engineering, Vol. 176, 804—807 (1953).
10. Symonds, P. S. and Neal, B. G., "Recent Progress in the Plastic Methods of Structural Analysis". J. Franklin Inst. Vol. 252, 383—407, 469—492 (1951).
11. Greenberg, H. J. and Prager, W., "Limit Design of Beams and Frames". Trans. A.S.C.E., Vol. 117, 447 (1952).
12. Haythornthwaite, R. M., "Behaviour in the Plastic Range of Laterally Loaded Structural Frameworks". Ph. D. Thesis, London University, England (1952). (2 Vols.)
13. Kuzmanović, "A Criticism of the Plastic-Hinge Concept Used in the Plastic Theory of Structures". Publ. of Int. Assn. for Bridge and Struct. Engineering, Vol. XIX, 147—154 (1959).
14. Yang, C. H., Beedle, L. S., Johnston, B. G., "Residual Stress and the Compressive Strength of Steel Beams". Welding Journal, Vol. 31 (4), 205 (1955).

15. Hendry, A. W., Jaeger, L. G., "The Analysis of Grid Frameworks and Related Structures". Chatto & Windus, London, 113—115 (1958).
16. Covington, B. J., "Analysis of Interconnected Girders and Grillage Beams". Ph. D. Dissertation, Northwestern University, Evanston, Illinois (1955).
17. Reddy, D. V., Hendry, A. W., "A Rapid Moment and Torque Distribution Method for Grid Framework Analysis". Civil Engineering and Public Works Review, Vol. 53, No. 637, 867—872 (1959).
18. Lightfoot, E., Sawko, F., "Structural Frame Analysis by Electronic Computer". Engineering, Vol. 187, No. 4843, 18—20 (1959).

Summary

A method of elasto-plastic analysis is presented for the load-deflection behaviour of an interconnected beam system without torsion. Owing to the importance of the stiffness criterion in plastic theory the general approach has been to express incremental loading as a function of the displacement at the load point.

The possible modes of elasto-plastic behaviour are indicated and Moment ratios examined to decide on the particular mode. A step-by-step procedure is used: as soon as a plastic hinge is formed it is replaced by a mechanical hinge and the resulting transformed structure is analysed for a further increment in deflection at the load point. The moments and deflections are obtained as functions of the incremental deflections and a flexural stiffness parameter.

The analytical solutions are confirmed by a satisfactory measure of agreement between the theoretical values and experimental results on a model grid framework tested to collapse.

Résumé

Les auteurs exposent dans cet article une méthode de calcul élasto-plastique, qui a pour but l'étude de la relation existant entre la grandeur de la charge appliquée et la déformation d'une grille de poutres, en négligeant l'influence de la torsion. Conformément à l'influence du critère de rigidité dans la théorie de la plasticité, le procédé général consiste à exprimer l'augmentation de la charge en fonction du déplacement de son point d'application.

Les auteurs énumèrent les différentes possibilités du comportement élasto-plastique et ils indiquent pour chaque forme le rapport correspondant des moments. Il s'agit d'un procédé par cheminement: aussitôt qu'une articulation plastique est formée, elle est remplacée par une articulation mécanique et l'on calcule l'augmentation du déplacement du point d'application de la charge pour ce nouveau système. Les moments et les flèches peuvent être déterminés en fonction de l'augmentation du déplacement et d'un paramètre de rigidité.

Les solutions analytiques concordent de façon satisfaisante avec les résultats obtenus par des essais, effectués sur un modèle de grille de poutres, qui a été chargé jusqu'à la rupture.

Zusammenfassung

Für das Belastungs-Durchbiegungsverhalten eines Trägerrostes ohne Torsionseinfluß wird eine elasto-plastische Untersuchungsmethode dargestellt. Entsprechend der Wichtigkeit des Steifigkeitskriteriums in der Plastizitätstheorie bestand das allgemeine Vorgehen darin, einen Belastungszuwachs als Funktion der Verschiebung des belasteten Punktes auszudrücken.

Die verschiedenen Formen des elasto-plastischen Verhaltens werden angegeben und die Verhältnisse der Momente, die zu einer bestimmten Form führen, werden untersucht. Das Verfahren wird schrittweise durchgeführt: sobald sich ein Fließgelenk ausgebildet hat, wird es durch ein mechanisches Gelenk ersetzt, und das damit geschaffene neue Tragsystem wird für einen weiteren Durchbiegungszuwachs im belasteten Punkt untersucht. Die Momente und Durchbiegungen lassen sich als Funktionen des Durchbiegungszuwachses und eines Steifigkeitsparameters bestimmen.

Die analytischen Lösungen werden bestätigt durch eine befriedigende Übereinstimmung zwischen den theoretischen Werten und den versuchstechnischen Resultaten an einem Trägerrostmodell, das bis zum Bruch untersucht wurde.