Curved edge disturbances in circular cylindrical shells

Autor(en): **Holand, Ivar**

Objekttyp: **Article**

Zeitschrift: **IABSE publications = Mémoires AIPC = IVBH Abhandlungen**

Band (Jahr): **20 (1960)**

PDF erstellt am: **01.06.2024**

Persistenter Link: https://doi.org/10.5169/seals-17556

Curved Edge Disturbances in Circular Cylindrical Shells

Perturbations marginales sur le bord incurvé dans les voiles cylindriques circulaires

Randstörungen am gekrümmten Rand in Kreiszylinderschalen

IVAR HOLAND
Dr. techn., Technical University of Norway, Trondheim

Introduction

In a paper [1] published in the 1959 volume of "Publications" the author has shown an analytical treatment of the present problem based on DONNELL's theory [3] for circular cylindrical shells. The treatment led up to closed formulas for edge value matrices and damping matrices. The simplifications made in DONNELL's theory will cause a reduced accuracy when displacements and forces are characterized by a small number of waves along the circumference. The present paper shows how simple correction terms may be found, which eliminate all significant errors.

The paper constitutes a supplement to reference [1], and to avoid an unnecessary repetition of the reasoning, the present paper will to a large extent refer to [1].

1. Basic Theory

In [2] the author has shown that the stress resultants and displacements of a cylindrical shell may be expressed approximately by the displacement w normal to the shell surface as follows (compare the notation list at the end of the paper):

$$M_x = \frac{K}{R^2}\left[\frac{\partial^2 w}{\partial \xi^2} + \nu\left(\frac{\partial^2 w}{\partial \varphi^2} + w\right)\right], \quad \text{(a)}$$

$$M_\varphi = \frac{K}{R^2}\left[\frac{\partial^2 w}{\partial \varphi^2} + w + \nu\frac{\partial^2 w}{\partial \xi^2}\right], \quad \text{(b)} \qquad (1)$$

$$
\begin{aligned}
Q_x &= \frac{K}{R^3}\frac{\partial}{\partial \xi}(\nabla^2 w + w), && \text{(c)}\\
Q_\varphi &= \frac{K}{R^3}\frac{\partial}{\partial \varphi}(\nabla^2 w + w), && \text{(d)}\\
R_x &= \frac{K}{R^3}\frac{\partial}{\partial \xi}\left[\frac{\partial^2 w}{\partial \xi^2} + (2-\nu)\left(\frac{\partial^2 w}{\partial \varphi^2} + w\right)\right], && \text{(e)}\\
R_\varphi &= \frac{K}{R^3}\frac{\partial}{\partial \varphi}\left[(2-\nu)\frac{\partial^2 w}{\partial \xi^2} + \left(\frac{\partial^2 w}{\partial \varphi^2} + w\right)\right], && \text{(f)} \quad (1)\\
N_\varphi &= -\frac{K}{R^3}\left(\nabla^4 w + \frac{\partial^2 w}{\partial \varphi^2}\right), && \text{(g)}\\
\frac{\partial}{\partial \xi} N_{x\varphi} &= \frac{K}{R^3}\frac{\partial}{\partial \varphi}\left(\nabla^4 w + 2\frac{\partial^2 w}{\partial \varphi^2}\right), && \text{(h)}\\
\frac{\partial^2}{\partial \xi^2} N_x &= -\frac{K}{R^3}\frac{\partial^2}{\partial \varphi^2}\left(\nabla^4 w + 2\frac{\partial^2 w}{\partial \varphi^2}\right), && \text{(i)}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^3 u}{\partial \xi^3} &= -\frac{K}{E h R^2}\nabla^4\left[\frac{\partial^2 w}{\partial \varphi^2} - \nu\frac{\partial^2 w}{\partial \xi^2} + 2w\right],\\
\frac{\partial^4 v}{\partial \xi^4} &= \frac{K}{E h R^2}\nabla^4\frac{\partial}{\partial \varphi}\left[\frac{\partial^2 w}{\partial \varphi^2} + (2+\nu)\frac{\partial^2 w}{\partial \xi^2} + 2w\right].
\end{aligned} \qquad (2)
$$

The angular rotation in the direction of the arc and the torsional moment are given by the known expressions

$$
\begin{aligned}
\vartheta_\varphi &= \frac{1}{R}\left(\frac{\partial w}{\partial \varphi} - v\right), && \text{(a)}\\
M_{x\varphi} &= \frac{K}{R^2}(1-\nu)\frac{\partial}{\partial \xi}\left(\frac{\partial w}{\partial \varphi} - v\right). && \text{(b)}
\end{aligned} \qquad (3)
$$

Finally the angular rotation in the direction of the axis is simply

$$
\vartheta_x = \frac{1}{R}\frac{\partial w}{\partial \xi}. \qquad (4)
$$

With approximations corresponding to those in (1) and (2) the differential equation of the shell is

$$
\nabla^8 w + 2\frac{\partial^6 w}{\partial \varphi^6} + 4c^4\frac{\partial^4 w}{\partial \xi^4} = 0. \qquad (5)
$$

The expressions given above will be reduced to those used in [1] when the secondary terms w in (1a–f) and (2), $\frac{\partial^2 w}{\partial \varphi^2}$ in (1g–i), v in (3) and $\frac{\partial^6 w}{\partial \varphi^6}$ in (5) are neglected.

The accuracy of the expressions above in relation to FLÜGGE's theory [4] may be expressed mathematically as follows:

Imagine the main terms of an expression to be $\frac{\partial^4 w}{\partial \xi^4}$ and $\frac{\partial^4 w}{\partial \varphi^4}$. In comparison with these terms the secondary terms w and $\frac{\partial^2 w}{\partial \xi^2}$ have been neglected. The secondary terms $\frac{\partial^2 w}{\partial \varphi^2}$, however, are retained in contradistinction to the approximations made in DONNELL's theory. A more accurate investigation shows that the inclusion of these secondary terms provides an accuracy which in practically every case of thin shells shows a negligible difference from that obtained from the complete theory of FLÜGGE.

2. The Characteristic Equation

Proceeding in the usual manner by expressing the displacement w by a series

$$w = \sum_{m=1}^{\infty} C e^{\lambda \xi} \sin m \varphi, \tag{6}$$

one obtains (from (5)) the characteristic equation

$$(\lambda^2 - m^2)^4 - 2 m^6 + 4 c^4 \lambda^4 = 0. \tag{7}$$

The assumptions made above, namely

$$|w| + \left|\frac{\partial^2 w}{\partial \xi^2}\right| \ll \left|\frac{\partial^4 w}{\partial \xi^4}\right| + \left|\frac{\partial^4 w}{\partial \varphi^4}\right|,$$

give for each term of the series (6)

$$1 + \lambda^2 \ll \lambda^4 + m^4.$$

Hence, within the limits of accuracy, Eq. (7) may be replaced by

$$[\lambda^2 - (m^2 - \tfrac{1}{2})]^4 + 4 c^4 \lambda^4 = 0. \tag{8}$$

This equation may be solved explicitly in the manner discribed in [1] and gives the eight roots

$$\hat{\lambda} = \pm(\alpha_1 \pm i \beta_1), \qquad \hat{\lambda} = \pm(\alpha_2 \pm i \beta_2), \tag{9}$$

where $\hat{\lambda}$ is the reduced root

$$\hat{\lambda} = \frac{\lambda}{c} \tag{10}$$

and furthermore

$$\begin{aligned} \alpha_1 &= \tfrac{1}{2}\{\ \ 1 + [(1+\epsilon^2)^{1/2} + \epsilon]^{1/2}\}, & \beta_1 &= \tfrac{1}{2}\{1 + [(1+\epsilon^2)^{1/2} - \epsilon]^{1/2}\}, \\ \alpha_2 &= \tfrac{1}{2}\{-1 + [(1+\epsilon^2)^{1/2} + \epsilon]^{1/2}\}, & \beta_2 &= \tfrac{1}{2}\{1 - [(1+\epsilon^2)^{1/2} - \epsilon]^{1/2}\}, \end{aligned} \tag{11}$$

where

$$\epsilon = \frac{2 m^2}{c^2}\left(1 - \frac{1}{2 m^2}\right). \tag{12}$$

The roots (11) are formally identical with the roots (17) in [1]. The only difference appears in the parameter ϵ.

As the principal roots are chosen again

$$\hat{\lambda}_1 = -\alpha_1 + i\beta_1, \qquad \hat{\lambda}_2 = -\alpha_2 + i\beta_2. \tag{13}$$

It may be noted that the root $\hat{\lambda}_1$ is the same as the one given by BERGER [5] in a somewhat different form, as an approximate solution of Flügge's characteristic equation.

From the formulas (11) is found that the root $\hat{\lambda}_1$ is of the order of magnitude 1 and the root $\hat{\lambda}_2$ is of the order of magnitude ϵ for small values of m, say $m < c$, that is $\epsilon < 2$. For $\epsilon > 2$ the following relations hold approximately (compare [1] Fig. 6)

$$\begin{aligned} \alpha_1 &\approx \tfrac{1}{2}(1 + \sqrt{2\epsilon}), & \beta_1 &\approx \tfrac{1}{2}\left(1 + \frac{1}{\sqrt{2\epsilon}}\right), \\ \alpha_2 &\approx \tfrac{1}{2}(-1 + \sqrt{2\epsilon}), & \beta_2 &\approx \tfrac{1}{2}\left(1 - \frac{1}{\sqrt{2\epsilon}}\right). \end{aligned} \tag{14}$$

Hence, the terms neglected in the approximate theory presented here are of the relative magnitude

$$\frac{1}{c^2} = \frac{1}{\sqrt{3(1-v^2)}}\,\frac{h}{R} \quad \text{and} \quad \frac{1}{m^4}.$$

The first of these terms is unsignificant in every case of thin shells. The term $1/m^4$ may seem to be of some importance when m is a small number. As the weakly damped waves corresponding to $\hat{\lambda}_2$ in case of $m = 1$ are replaced by a polynomial function, it should be remembered that this case must be treated separately. For $m \geqq 2$ the term $1/m^4$ will be of very little importance in actual design. If desired, expressions which are more accurate for small values of m may be found in [6].

3. Characteristic Coefficients (Coefficients to the Constants of Integration)

As was shown in [1] each term of the series which constitutes the solution of Eq. (5), may be written

$$w = \sin(m\varphi)\,\mathrm{R}\{C_1 e^{\lambda_1 \xi} + C_2 e^{\lambda_2 \xi}\}. \tag{15}$$

In (15) the damped waves only are included. By using Eqs. (1)—(4) one obtains expressions corresponding to (15) for all statical quantities. For an arbitrarily chosen quantity H the expression may be written in the form

$$H = [H]\,\mathrm{R}\{C_1 \hat{H}_1 e^{\lambda_1 \xi} + C_2 \hat{H}_2 e^{\lambda_2 \xi}\}. \tag{16}$$

The multipliers $[H]$, which contain the trigonometric function of φ and quantities depending on the dimensions of the shell, will be the same as those given in [1]. The characteristic coefficients $\hat{H}_1$ and $\hat{H}_2$ will, however, contain correction terms resulting from the additional terms as well in the parameter ϵ as in the expressions (1)—(4). As an example of the derivations needed the expression $\nabla^2 w$ may be considered. From (15) is obtained

$$\nabla^2 w = \frac{\partial^2 w}{\partial \xi^2} + \frac{\partial^2 w}{\partial \varphi^2} = \sin(m\varphi)\, c^2 \mathrm{R}\left\{C_1\left(\hat{\lambda}_1^2 - \frac{m^2}{c^2}\right)e^{\lambda_1 \xi} + C_2\left(\hat{\lambda}_2^2 - \frac{m^2}{c^2}\right)e^{\lambda_2 \xi}\right\}. \quad (17)$$

By use of (11) and (12)

$$\begin{aligned}\hat{\lambda}_1^2 - \frac{m^2}{c^2} &= \frac{\epsilon}{2} + (\alpha_1 - \beta_1) - i(\alpha_1 + \beta_1) - \frac{m^2}{c^2}\\ &= \alpha_1 - \beta_1 - i(\alpha_1 + \beta_1) - \frac{1}{2c^2}.\end{aligned}$$

Here, the term $\frac{1}{2c^2}$ may be neglected, as $\alpha_1 + \beta_1 > 2$. Hence,

$$\hat{\lambda}_1^2 - \frac{m^2}{c^2} = \alpha_1 - \beta_1 - i(\alpha_1 + \beta_1). \quad (18)$$

In the same manner

$$\hat{\lambda}_2^2 - \frac{m^2}{c^2} = -(\alpha_1 - \beta_1) - i(\alpha_2 - \beta_2) - \frac{1}{2c^2}.$$

In this case the last term cannot be neglected, as it is of the same order of magnitude as the first, when m is small. However, the expression may be approximated by

$$\hat{\lambda}_2^2 - \frac{m^2}{c^2} = -[(\alpha_1 - \beta_1) + i(\alpha_2 - \beta_2)]\left(1 + \frac{1}{2m^2}\right). \quad (19)$$

For $m > c$ Eq. (19) is obviously within the limits of accuracy. For $m < c$ one finds

$$\alpha_1 - \beta_1 \approx \frac{\epsilon}{2} - \frac{\epsilon^3}{16},$$

which gives $(\alpha_1 - \beta_1)\left(1 + \frac{1}{2m^2}\right) \approx (\alpha_1 - \beta_1) + \frac{\epsilon}{2}\frac{1}{2m^2} - \frac{\epsilon^3}{16 \cdot 2m^2}$

$$\approx (\alpha_1 - \beta_1) + \frac{1}{2c^2}$$

and furthermore

$$(\alpha_2 - \beta_2)\frac{1}{2m^2} \approx \frac{\epsilon^2}{8}\frac{1}{2m^2} \approx \epsilon\frac{1}{8c^2},$$

which is negligible compared with $\alpha_1 - \beta_1 \approx \epsilon/2$. Hence, Eq. (19) is within the stipulated limits of accuracy in all cases.

The resulting characteristic coefficients are given in Table 1. The new notation

Table 1. Multipliers and Characteristic Coefficients

Quantity H	Multiplier $[H]$	$\hat{H}_1$	$\hat{H}_2$
N_x	$\sin m\varphi$	$-i$	$i\left(1-\frac{1}{m^2}\right)$
N_φ	$2\frac{c^2}{m^2}\sin m\varphi$	$\frac{1}{2}(\alpha_1+\beta_1)+i[b+\frac{1}{2}(\alpha_1-\beta_1)]$	$-\frac{1}{2}(\alpha_2-\beta_2)-i[b-\frac{1}{2}(\alpha_2+\beta_2)]$
$N_{x\varphi}$	$\frac{c}{m}\cos m\varphi$	$\beta_1+i\,\alpha_1$	$-(\beta_2+i\,\alpha_2)\left(1-\frac{1}{m^2}\right)$
M_x	$\frac{R}{m^2}\sin m\varphi$	$(1-\nu)\,b+\frac{1}{2}(\alpha_1-\beta_1)-i\,\frac{1}{2}(\alpha_1+\beta_1)$	$[(1-\nu)\,b-\frac{1}{2}(\alpha_2+\beta_2)-i\,\frac{1}{2}(\alpha_2-\beta_2)]\left(1-\frac{1}{2m^2}\right)$
M_φ	$\frac{R}{m^2}\sin m\varphi$	$-b+\nu[b+\frac{1}{2}(\alpha_1-\beta_1)-i\,\frac{1}{2}(\alpha_1+\beta_1)]$	$\{-b+\nu[b-\frac{1}{2}(\alpha_2+\beta_2)-i\,\frac{1}{2}(\alpha_2-\beta_2)]\}\left(1-\frac{1}{2m^2}\right)$
$M_{x\varphi}$	$\frac{1-\nu}{2}\,\frac{R}{mc}\cos m\varphi$	$-\alpha_1+i\,\beta_1$	$-(\alpha_2-i\,\beta_2)\left(1-\frac{1}{m^2}\right)$
Q_x	$\frac{c}{m^2}\sin m\varphi$	$-b+\beta_1+i\,(b+\alpha_1)$	$[b-\beta_2+i\,(b-\alpha_2)]\left(1-\frac{1}{2m^2}\right)$
Q_φ	$\frac{1}{m}\cos m\varphi$	$\frac{1}{2}(\alpha_1-\beta_1)-i\,\frac{1}{2}(\alpha_1+\beta_1)$	$[-\frac{1}{2}(\alpha_2+\beta_2)-i\,\frac{1}{2}(\alpha_2-\beta_2)]\left(1-\frac{1}{2m^2}\right)$
R_x	$\frac{c}{m^2}\sin m\varphi$	$\beta_1+b\,(\alpha_2-\nu\,\alpha_1)+i[\alpha_1+b\,(\beta_2+\nu\,\beta_1)]$	$\{-\beta_2+b\,(\alpha_1-\nu\,\alpha_2)-i[\alpha_2-b\,(\beta_1+\nu\,\beta_2)]\}\left(1-\frac{1}{2m^2}\right)$
R_φ	$\frac{1}{m}\cos m\varphi$	$(1-\nu)\,b+(2-\nu)[\frac{1}{2}(\alpha_1-\beta_1)-i\,\frac{1}{2}(\alpha_1+\beta_1)]$	$\{(1-\nu)\,b-(2-\nu)[\frac{1}{2}(\alpha_2+\beta_2)-i\,\frac{1}{2}(\alpha_2-\beta_2)]\}\left(1-\frac{1}{2m^2}\right)$
u	$\frac{R}{Eh}\,\frac{2c^3}{m^4}\sin m\varphi$	$[-\beta_2+\nu\,\beta_1+i\,(\alpha_2+\nu\,\alpha_1)]\,b\left(1+\frac{1}{2m^2}\right)$	$[\beta_1-\nu\,\beta_2-i\,(\alpha_1+\nu\,\alpha_2)]\,b$
v	$\frac{R}{Eh}\,\frac{2c^2}{m^3}\cos m\varphi$	$\{-\frac{1}{2}(\alpha_2-\beta_2)-i[(1+\nu)\,b+\frac{1}{2}(\alpha_2+\beta_2)]\}\left(1+\frac{1}{2m^2}\right)$	$\frac{1}{2}(\alpha_1+\beta_1)+i[(1+\nu)\,b-\frac{1}{2}(\alpha_1-\beta_1)]$
w	$\frac{R}{Eh}\,\frac{2c^2}{m^2}\sin m\varphi$	1	1
ϑ_x	$\frac{1}{Eh}\,\frac{2c^3}{m^2}\sin m\varphi$	$-\alpha_1+i\,\beta_1$	$-\alpha_2+i\,\beta_2$
ϑ_φ	$\frac{1}{Eh}\,\frac{2c^2}{m}\cos m\varphi$	1	$1-\frac{1}{m^2}$

$$b = \frac{\epsilon}{4} = \frac{m^2}{2c^2}\left(1-\frac{1}{2m^2}\right)$$

is used in the table. If the terms containing $\frac{1}{m^2}$ are neglected, the coefficients will be reduced to those given in [1].

4. Edge Value Relations and Damping Relations in Closed Form

All quantities are now given by Eq. (16), which contains the unknown constants of integration. As was explained in [1] it is more advantageous to eliminate the constants of integration and express all quantities by the reduced edge quantities

$$\hat{N}_{x0}, \quad \hat{N}_{x\varphi 0}, \quad \hat{\vartheta}_{x0} \quad \text{and} \quad \hat{w}_0.$$

Let

$$C_1 = A_1 + i\,B_1, \qquad C_2 = A_2 + i\,B_2.$$

From Eq. (16) and Table 1 the following expressions are obtained for the edge quantities mentioned (in matrix notation)

$$\begin{bmatrix} \hat{N}_{x0} \\ \hat{N}_{x\varphi 0} \\ \hat{\vartheta}_{x0} \\ \hat{w}_0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & -\left(1-\frac{1}{m^2}\right) \\ \beta_1 & -\alpha_1 & -\beta_2\left(1-\frac{1}{m^2}\right) & \alpha_2\left(1-\frac{1}{m^2}\right) \\ -\alpha_1 & -\beta_1 & -\alpha_2 & -\beta_2 \\ 1 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} A_1 \\ B_1 \\ A_2 \\ B_2 \end{bmatrix}. \tag{20}$$

These equations may be solved for A_1, B_1, A_2 and B_2, giving

$$\begin{bmatrix} A_1 \\ B_1 \\ A_2 \\ B_2 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} \left(\alpha_2+\beta_2+\frac{b}{m^2}\right) & 1 & -1 & -\left(\alpha_2-\beta_2+\frac{b}{m^2}\right) \\ -\left(\alpha_2-\beta_2-\frac{b}{m^2}\right) & -1 & -1 & -\left(\alpha_2+\beta_2-\frac{b}{m^2}\right) \\ -\left(\alpha_1-\beta_1+\frac{b}{m^2}\right) & -1 & 1 & (\alpha_1+\beta_1) \\ -\left(\alpha_1+\beta_1+\frac{2}{m^2}\right) & -\left(1+\frac{1}{m^2}\right) & -\left(1+\frac{1}{m^2}\right) & -\left(\alpha_1-\beta_1+\frac{b}{m^2}\right) \end{bmatrix} \begin{bmatrix} \hat{N}_{x0} \\ \hat{N}_{x\varphi 0} \\ \hat{\vartheta}_{x0} \\ \hat{w}_0 \end{bmatrix}. \tag{21}$$

Substitution of these constants of integration in Eq. (16) yields the value of any statical quantity at an arbitrarily chosen section expressed by the edge values $\hat{N}_{x0}$, $\hat{N}_{x\varphi 0}$, $\hat{\vartheta}_{x0}$ and $\hat{w}_0$. Let S be the column vector

$$S = \{\hat{N}_x,\ \hat{N}_{x\varphi},\ \hat{\vartheta}_x,\ \hat{w}\}, \tag{22}$$

and let the edge value of S be denoted S_0. The substitution mentioned will then lead to an expression in the form

$$S = [s_{ik}]\,S_0, \tag{23}$$

where $[s_{ik}]$ is a $4 \cdot 4$ matrix expressing the damping of the vector S. The elements s_{ik} are functions of the coordinate ξ. The procedure described leads, after rather tedious developments, to the following expressions for the matrix elements

$$\begin{bmatrix} s_{11} \\ s_{21} \\ s_{31} \\ s_{41} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} -(\alpha_2-\beta_2) & (\alpha_2+\beta_2) & (\alpha_1+\beta_1) & (\alpha_1-\beta_1) \\ 2b & -2b\left(1+\frac{1}{m^2}\right) & -2b & -2b \\ -2b\left(1+\frac{1}{m^2}\right) & -2b & 2b\left(1+\frac{1}{m^2}\right) & -2b\left(1+\frac{1}{m^2}\right) \\ (\alpha_2+\beta_2) & (\alpha_2-\beta_2) & -(\alpha_1-\beta_1) & (\alpha_1+\beta_1)\left(1+\frac{1}{m^2}\right) \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{bmatrix}, \tag{24a}$$

$$\begin{bmatrix} s_{12} \\ s_{22} \\ s_{32} \\ s_{42} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} -1 & 1 & 1 & \left(1-\frac{1}{m^2}\right) \\ (\alpha_1+\beta_1) & -(\alpha_1-\beta_1) & -(\alpha_2-\beta_2) & -(\alpha_2+\beta_2) \\ -(\alpha_1-\beta_1) & -(\alpha_1+\beta_1) & (\alpha_2+\beta_2) & -(\alpha_2-\beta_2) \\ 1 & 1 & -1 & \left(1+\frac{1}{m^2}\right) \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{bmatrix}, \tag{24b}$$

$$\begin{bmatrix} s_{13} \\ s_{23} \\ s_{33} \\ s_{43} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} -1 & -1 & 1 & -\left(1-\frac{1}{m^2}\right) \\ (\alpha_1-\beta_1) & (\alpha_1+\beta_1) & -(\alpha_2+\beta_2) & (\alpha_2-\beta_2) \\ (\alpha_1+\beta_1) & -(\alpha_1-\beta_1) & -(\alpha_2-\beta_2) & -(\alpha_2+\beta_2) \\ -1 & 1 & 1 & \left(1+\frac{1}{m^2}\right) \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{bmatrix}, \tag{24c}$$

$$\begin{bmatrix} s_{14} \\ s_{24} \\ s_{34} \\ s_{44} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} -(\alpha_2+\beta_2) & -(\alpha_2-\beta_2) & (\alpha_1-\beta_1) & -(\alpha_1+\beta_1)\left(1-\frac{1}{m^2}\right) \\ 2b\left(1-\frac{1}{m^2}\right) & 2b & -2b\left(1-\frac{1}{m^2}\right) & 2b\left(1-\frac{1}{m^2}\right) \\ 2b & -2b\left(1-\frac{1}{m^2}\right) & -2b & -2b \\ -(\alpha_2-\beta_2) & (\alpha_2+\beta_2) & (\alpha_1+\beta_1) & (\alpha_1-\beta_1) \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{bmatrix} \tag{24d}$$

where

$$\begin{aligned} f_1 &= e^{-\alpha_1 c\xi} \cos\beta_1 c\xi, & f_3 &= e^{-\alpha_2 c\xi} \cos\beta_2 c\xi, \\ f_2 &= e^{-\alpha_2 c\xi} \sin\beta_1 c\xi, & f_4 &= e^{-\alpha_2 c\xi} \sin\beta_2 c\xi. \end{aligned} \tag{25}$$

When the value of S at an arbitrarily chosen arc is known, the other quantities may be calculated from S by relations independent of ξ as was explained in [1]. This is obtained by using Eq. (16) to express the remaining quantities at the section $\xi = 0$ by S_0. As the origin of ξ has been chosen arbitrarily, the relations obtained are valid not at $\xi = 0$ only, but also at an arbitrarily chosen arc. The resulting expressions may be written

$$\begin{bmatrix} \hat{u} \\ \hat{v} \\ -\hat{M}_x \\ \hat{R}_x \end{bmatrix} = \begin{bmatrix} -(\alpha_1+\alpha_2)\,b\left(1+\frac{1}{m^2}\right) & -(1-\nu)\,b\left(1+\frac{1}{2m^2}\right) & -\frac{1}{2m^2}\,b & (\beta_1-\beta_2)\,b \\ -(1-\nu)\,b\left(1+\frac{1}{2m^2}\right) & -\frac{1}{2}(\alpha_1+\alpha_2) & \frac{1}{2}(\beta_1-\beta_2) & 1 \\ -\frac{1}{2m^2}\,b & \frac{1}{2}(\beta_1-\beta_2) & \frac{1}{2}(\alpha_1+\alpha_2) & (1+\nu)\,b\left(1-\frac{1}{2m^2}\right) \\ (\beta_1-\beta_2)\,b & 1 & (1+\nu)\,b\left(1-\frac{1}{2m^2}\right) & (\alpha_1+\alpha_2)\,b\left(1-\frac{1}{m^2}\right) \end{bmatrix} \begin{bmatrix} \hat{N}_x \\ \hat{N}_{x\varphi} \\ \hat{\vartheta}_x \\ \hat{w} \end{bmatrix}, \tag{26}$$

$$\begin{bmatrix} \hat{N}_\varphi \\ \hat{M}_\varphi \\ \hat{R}_\varphi \\ \hat{\vartheta}_\varphi \\ \hat{Q}_x \\ \hat{Q}_\varphi \\ \hat{M}_{x\varphi} \end{bmatrix} = \begin{bmatrix} b\left(1+\frac{1}{2m^2}\right) & \frac{1}{2}(\alpha_1+\alpha_2) & -\frac{1}{2}(\beta_1-\beta_2) & -\frac{1}{2m^2}\,b \\ \frac{\nu}{2m^2}\,b & -\frac{\nu}{2}(\beta_1-\beta_2) & -\frac{\nu}{2}(\alpha_1+\alpha_2) & -(1+\nu)\,b\left(1-\frac{1}{2m^2}\right) \\ \left(1-\frac{\nu}{2}\right)\frac{1}{m^2}\,b & -\left(1-\frac{\nu}{2}\right)(\beta_1-\beta_2) & -\left(1-\frac{\nu}{2}\right)(\alpha_1+\alpha_2) & -(3-\nu)\,b\left(1-\frac{1}{2m^2}\right)^{1)} \\ 0 & \frac{1}{2m^2} & -\frac{1}{2m^2} & 1-\frac{1}{m^2} \\ (\beta_1-\beta_2)\,b & 1 & 2b & (\alpha_1+\alpha_2)\,b\left(1-\frac{1}{m^2}\right) \\ \frac{1}{2m^2}\,b & -\frac{1}{2}(\beta_1-\beta_2) & -\frac{1}{2}(\alpha_1+\alpha_2) & -2b\left(1-\frac{1}{2m^2}\right) \\ -\frac{1}{m^2}\,b & 0 & 1 & \frac{1}{m^2}\,b \end{bmatrix} \begin{bmatrix} \hat{N}_x \\ \hat{N}_{x\varphi} \\ \hat{\vartheta}_x \\ \hat{w} \end{bmatrix}. \tag{27}$$

---

1) The corresponding term in [1] Eq. (47) is erroneously given as $-4\left(1-\frac{\nu}{2}\right)b$ instead of $-(3-\nu)b$.

Now all quantities are given expressed by the edge quantities S_0, and any problem concerning end disturbances may be solved as outlined in [1].

The roots of the characteristic equation occur in all the formulas given. The quantities α_1 and β_1 are tabulated for values of b from 0.00 to 1.00 in Table 2. α_2 and β_2 are found from

$$\alpha_2 = \alpha_1 - 1, \qquad \beta_2 = 1 - \beta_1.$$

For values of $b > 1$ the approximate formulas (14) may be used. The error they give is for $b = 1$ 0,6 % in α_1, and 0,2 % in β_1.

Table 2

b	α_1	β_1	b	α_1	β_1
0.00	1.0000	1.0000	0.17	1.1873	0.8638
	101	–99		113	–59
0.01	1.0101	0.9901	0.18	1.1986	0.8579
	103	–97		113	–57
0.02	1.0204	0.9804	0.19	1.2099	0.8522
	104	–95		113	–55
0.03	1.0308	0.9709	0.20	1.2212	0.8467
	106	–92		112	–53
0.04	1.0414	0.9617	0.21	1.2324	0.8414
	108	–90		112	–52
0.05	1.0522	0.9527	0.22	1.2436	0.8362
	109	–87		112	–50
0.06	1.0631	0.9440	0.23	1.2548	0.8312
	110	–85		111	–48
0.07	1.0741	0.9355	0.24	1.2659	0.8264
	111	–83		110	–46
0.08	1.0852	0.9272	0.25	1.2769	0.8218
	112	–80		109	–45
0.09	1.0964	0.9192	0.26	1.2878	0.8173
	113	–78		109	–43
0.10	1.1077	0.9114	0.27	1.2987	0.8130
	113	–75		108	–42
0.11	1.1190	0.9039	0.28	1.3095	0.8088
	113	–73		108	–40
0.12	1.1303	0.8966	0.29	1.3203	0.8048
	114	–70		107	–39
0.13	1.1417	0.8896	0.30	1.3310	0.8009
	114	–68		106	–38
0.14	1.1531	0.8828	0.31	1.3416	0.7971
	114	–66		105	–37
0.15	1.1645	0.8762	0.32	1.3521	0.7934
	114	–63		105	–36
0.16	1.1759	0.8699	0.33	1.3626	0.7898
	114	–61		103	–34
0.17	1.1873	0.8638	0.34	1.3729	0.7864

b	α_1	β_1	b	α_1	β_1
0.34	1.3729	0.7864	0.57	1.5920	0.7289
	103	−34		87	−18
0.35	1.3832	0.7830	0.58	1.6007	0.7271
	103	−32		87	−18
0.36	1.3935	0.7798	0.59	1.6094	0.7253
	101	−31		86	−17
0.37	1.4036	0.7767	0.60	1.6180	0.7236
	101	−31		86	−17
0.38	1.4137	0.7736	0.61	1.6266	0.7219
	100	−30		85	−17
0.39	1.4237	0.7706	0.62	1.6351	0.7202
	99	−28		85	−16
0.40	1.4336	0.7678	0.63	1.6436	0.7186
	99	−28		84	−16
0.41	1.4435	0.7650	0.64	1.6520	0.7170
	98	−28		84	−15
0.42	1.4533	0.7622	0.65	1.6604	0.7155
	97	−26		83	−16
0.43	1.4630	0.7596	0.66	1.6687	0.7139
	97	−26		82	−15
0.44	1.4727	0.7570	0.67	1.6769	0.7124
	95	−25		82	−15
0.45	1.4822	0.7545	0.68	1.6851	0.7109
	95	−24		82	−14
0.46	1.4917	0.7521	0.69	1.6933	0.7095
	95	−24		81	−14
0.47	1.5012	0.7497	0.70	1.7014	0.7081
	94	−23		80	−14
0.48	1.5106	0.7474	0.71	1.7094	0.7067
	92	−23		80	−14
0.49	1.5198	0.7451	0.72	1.7174	0.7053
	93	−22		80	−13
0.50	1.5291	0.7429	0.73	1.7254	0.7040
	92	−21		79	−13
0.51	1.5383	0.7408	0.74	1.7333	0.7027
	91	−21		79	−13
0.52	1.5474	0.7387	0.75	1.7412	0.7014
	90	−21		78	−12
0.53	1.5564	0.7366	0.76	1.7490	0.7002
	90	−20		78	−13
0.54	1.5654	0.7346	0.77	1.7568	0.6989
	89	−19		77	−12
0.55	1.5743	0.7327	0.78	1.7645	0.6977
	89	−19		77	−12
0.56	1.5832	0.7308	0.79	1.7722	0.6965
	88	−19		77	−12
0.57	1.5920	0.7289	0.80	1.7799	0.6953

b	α_1	β_1	b	α_1	β_1
0.80	1.7799	0.6953	0.90	1.8543	0.6846
	76	−11		72	−10
0.81	1.7875	0.6942	0.91	1.8615	0.6836
	76	−12		72	−10
0.82	1.7951	0.6930	0.92	1.8687	0.6826
	75	−11		72	−9
0.83	1.8026	0.6919	0.93	1.8759	0.6817
	75	−11		71	−9
0.84	1.8101	0.6908	0.94	1.8830	0.6808
	75	−11		71	−10
0.85	1.8176	0.6897	0.95	1.8901	0.6798
	74	−10		70	−9
0.86	1.8250	0.6887	0.96	1.8971	0.6789
	74	−11		71	−9
0.87	1.8324	0.6876	0.97	1.9042	0.6780
	73	−10		70	−8
0.88	1.8397	0.6866	0.98	1.9112	0.6772
	73	−10		69	−9
0.89	1.8470	0.6856	0.99	1.9181	0.6763
	73	−10		69	−9
0.90	1.8543	0.6846	1.00	1.9250	0.6754

Notations

R	shell radius.
h	shell thickness.
$x, y, \xi = x/R, \varphi = y/R$	coordinates.
M_x, M_φ	bending moments.
$M_{x\varphi}$	torsional moment.
Q_x, Q_φ	transverse forces.
R_x, R_φ	resulting transverse edge forces.
N_x, N_φ	normal forces.
$N_{x\varphi}$	shear force.
$\epsilon_x, \epsilon_\varphi, \gamma_{x\varphi}$	unit deformations.
u, v, w	displacements in the directions x, φ and normal to the shell surface.
$\vartheta_x, \vartheta_\varphi$	angles of rotation.
E	modulus of elasticity.
ν	Poisson's ratio.
$K = E h^3/12(1-\nu^2)$	flexural rigidity.
$\nabla^2 = \partial^2/\partial\xi^2 + \partial^2/\partial\varphi^2$	Laplace's operator.
$\nabla^4 = \nabla^2\nabla^2$	differential operator.
$\nabla^8 = \nabla^4\nabla^4$	differential operator.

m	arbitrary number.
i	imaginary unit.
λ	root of the characteristic equation.
$\hat{\lambda}=\lambda/c$	reduced value of λ.
α, β	real and imaginary parts of $\hat{\lambda}$.
$\mathrm{R}\{z\}$	real part of a complex number z.
$b=\dfrac{m^2-\frac{1}{2}}{2c^2}$	dimensionless constant.
$\epsilon=4b$	dimensionless constant.
$c=\sqrt{R/h}\sqrt[4]{3(1-\nu^2)}$	dimensionless constant.
H	arbitrary statical quantity.
$[H]$	multiplier of H.
$\hat{H}$	reduced value of H.
H_0	edge value of H.
A, B, C	constants of integration.
f_1, f_2, f_3, f_4	damped trigonometric functions (see Eq. (25)).
$S=\{\hat{N}_x, \hat{N}_{x\varphi}, \hat{\vartheta}_x, \hat{w}\}$	column vector of statical quantities.
S_0	edge value of S.
$[s_{ik}]$	matrix defined by $S=[s_{ik}]S_0$.

References

1. Holand, I., "An Application of Donnell's Theory of Circular Cylindrical Shells to the Analysis of Curved Edge Disturbances". Intern. Assn. Bridge and Struct. Eng. Publications. Vol.19, Zürich 1959.
2. Holand, I., "A Contribution to the Theory of Cylindrical Shells". Second Symp. Concrete Shell Roof Constr., Teknisk Ukeblad, Oslo 1958.
3. Donnell, L. H., "Stability of Thin-Walled Tubes under Torsion", NACA Rep. No. 479, 1934.
4. Flügge, W., "Statik und Dynamik der Schalen". Springer, Berlin 1934.
5. Berger, E. R., "Die Auflösung der charakteristischen Gleichung für Zylinder-Schalen durch Iteration". B. u. St. 48, 62—65, 1953.
6. Holand, I., "Design of Circular Cylindrical Shells". Norges Tekniske Vitenskapsakademi, Series 2, No. 3, Oslo University Press, Oslo 1957.

Summary

The paper deals with the theory of curved edge disturbances in circular cylindrical shells and constitutes a supplement to a paper published by the author in the 1959 volume of "Publications". The present paper shows how a more accurate theory may be applied to the same problem, a theory which gives simple correction terms to the elements in edge value matrices and damping matrices given in the paper mentioned above. When these correction terms are included, the difference from Flügge's theory is insignificant. Tables are included to facilitate the use of the method.

Résumé

Cette contribution traite de la théorie des perturbations marginales sur le bord incurvé dans les voiles cylindriques et constitue un complément au travail publié par l'auteur en 1959 dans le 19e volume des «Mémoires». La présente contribution montre comment une théorie plus exacte peut être appliquée à la solution du même problème, théorie qui donne de simples termes correctifs des coefficients des matrices des valeurs marginales et des matrices qui décrivent l'amortissement des perturbations marginales, matrices qui sont contenues dans le travail mentionné ci-dessus. La différence, par rapport à la théorie de FLÜGGE, est insignifiante lorsqu'on tient compte de ces termes correctifs. Afin de faciliter l'application de cette méthode, des tables sont inclues à cette contribution.

Zusammenfassung

Dieser Artikel befaßt sich mit der Theorie der Randstörungen am gekrümmten Rand in Kreiszylinderschalen und bildet eine Ergänzung zu einer Arbeit, welche der Verfasser im neunzehnten Band der «Abhandlungen» (1959) veröffentlichte. Die vorliegende Abhandlung zeigt, wie eine genauere Theorie zur Lösung desselben Problems angewendet werden kann, eine Theorie, welche einfache Korrekturglieder liefert zu den in der oben erwähnten Arbeit enthaltenen Randwertmatrizen und den Matrizen, welche das Abklingen der Randstörung beschreiben.

Der Unterschied zur Theorie von FLÜGGE wird unbedeutend, falls diese Korrekturglieder in der Rechnung mitberücksichtigt werden. Um die Anwendung dieser Methode zu erleichtern, sind dem Artikel Tabellen beigefügt.