Stress and strain in thin shallow sperical calotte shells

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Stress and Strain in Thin Shallow Spherical Calotte Shells

Contraintes et allongements dans les calottes sphériques minces de faible hauteur

Spannung und Dehnung in dünnen, flachen Kugelkalottenschalen

Dr. Phil. GUNHARD-AESTIUS ORAVAS, Caracas, Venezuela [1])

Introduction

This paper is concerned with the development of an approximate solution applicable to the analysis of shallow thin segmental shells of spherical middle surface.

Such segmental or calotte shells have found frequent structural application on this continent and overseas [19, 22].

Stress evaluations of such shells have been either carried out by model analysis or by analytical methods which disregard the transverse bending stiffness of the shell [1, 5, 6, 7, 13, 14]. This lack of consideration of transverse stiffness in stress analysis may bring on a neglect of certain high stresses created by the boundary disturbances. The extent of this critical perturbation of the momentless state of stress depends largely upon the proportions of the shell, loading and its boundary contour.

For instance, a segmental dome over a triangular base exhibits a transverse bending zone which is comparable in surface area of the shell with the region where the momentless state of stress predominates. For such a shell a complete neglect of transverse bending in the stress analysis is not admissible. Such a thin shell behaves rather uneconomically stresswise because the salient feature of its deformation is a widespread transverse bending and not the desirable extensional state of strain.

A number of papers have been published on the analysis of spherical shells, which do not neglect its inherent transverse bending stiffness [2, 3 [2]), 4, 9,

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[1]) At present Sr. Structural Engineer, Smith, Hinchman & Grylls Associates, Inc., Detroit, Michigan.

[2]) This thesis has not been published in engineering literature and has been unavailable to the writer.

16, 18, 20, 21, 23, 24], though none of them have considered shallow thin calotte shells. In all these investigations transverse shear deformation of the shell has been suppressed as being of a negligibly small order of magnitude in comparison with the magnitude of bending and extensional deformations. This is assumed to be valid also for calotte shells. The present work seeks to solve the stress problem for thin shallow spherical shells covering polygonal type of base.

Formulation and Solution of Fundamental Differential Equations

Complementary Solution

The fundamental differential equations given by MUSHTARI [12] and VLASSOV [23] for thin shallow shells in generalized orthogonal curvilinear coordinates are

$$\begin{aligned} \nabla^4\Phi - E\,h\,\nabla_c^2 w &= -(1-\nu)\,\nabla^2\Gamma - E\,h\,\nabla^2(\epsilon\,T), \\ \nabla^4 W + \frac{1}{D}\nabla_c^2\Phi &= \frac{p_n}{D} - \frac{\Gamma}{D}(c_1+c_2),^{3)} \end{aligned} \tag{1}$$

where

Γ external load intensity potential function (see p_α and p_β below),

p_n, $p_\alpha = -\dfrac{\partial\Gamma}{A\,\partial\alpha}$, $p_\beta = -\dfrac{\partial\Gamma}{B\,\partial\beta}$ are components of load intensity (fig. 2),

w normal displacement function,

Φ stress function,

h constant shell thickness,

$D = E\,h^3/12\,(1-\nu^2)$ cylindrical bending rigidity,

ϵ linear thermal expansion,

$T = T\,(\alpha,\beta)$ temperature distribution function that describes the differential thermal increase or decrease from a stress and strain free temperature level of the shell bringing about an extension or contraction of shell's middle surface,

E Young's modulus,

ν Poisson's ratio,

and

$$\begin{aligned} \nabla^2 &= \frac{1}{A\,B}\left\{\frac{\partial}{\partial\alpha}\left[\frac{B}{A}\frac{\partial}{\partial\alpha}\right] + \frac{\partial}{\partial\beta}\left[\frac{A}{B}\frac{\partial}{\partial\beta}\right]\right\}, \\ \nabla_c^2 &= \frac{1}{A\,B}\left\{\frac{\partial}{\partial\alpha}\left[c_2\frac{B}{A}\frac{\partial}{\partial\alpha}\right] + \frac{\partial}{\partial\beta}\left[c_1\frac{A}{B}\frac{\partial}{\partial\beta}\right]\right\} \end{aligned}$$

and

$$\nabla^4 = \nabla^2\nabla^2.$$

[3]) For equivalent equations in "almost cartesian" coordinates, see «Zur Theorie der gekrümmten Platte großer Formänderung» by K. MARGUERRE, Proc. 5th Int. Congress Appl. Mech., 1938, p. 93.

In these formulas $A^2 = [A(\alpha,\beta)]^2$, and $B^2 = [B(\alpha,\beta)]^2$ represent coefficients of the fundamental metric of shell's middle surface and $c_1 = c_1(\alpha,\beta)$ and $c_2 = c_2(\alpha,\beta)$ are the curvatures of coordinate lines β = const. and α = const. respectively (fig. 1). Curvilinear coordinates α and β are taken to be coincident with the principal lines of curvature.

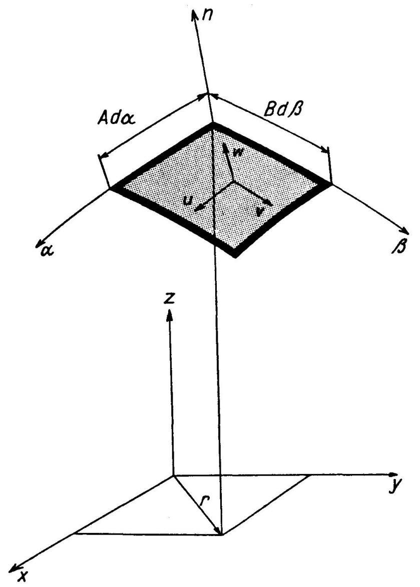

Fig. 1. Differential Element of Shell with Coordinates and Displacements.

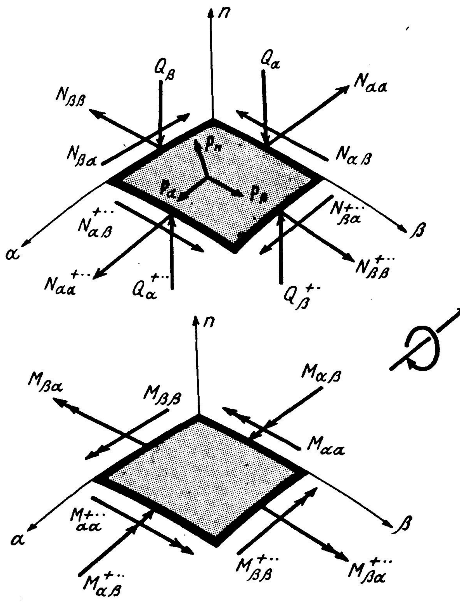

Fig. 2. Differential Element of Shell with Stress Couples, Stress Resultants and Surface Loads.

Sectional stress resultants and stress couples are conveniently expressed by the following relationships [8] (fig. 2):

$$
\begin{aligned}
N_{\alpha\alpha} &= \left\{\frac{1}{B}\frac{\partial}{\partial\beta}\left[\frac{1}{B}\frac{\partial\Phi}{\partial\beta}\right] + \frac{1}{A^2 B}\frac{\partial B}{\partial\alpha}\frac{\partial\Phi}{\partial\alpha}\right\},\\
N_{\beta\beta} &= \left\{\frac{1}{A}\frac{\partial}{\partial\alpha}\left[\frac{1}{A}\frac{\partial\Phi}{\partial\alpha}\right] + \frac{1}{A B^2}\frac{\partial A}{\partial\beta}\frac{\partial\Phi}{\partial\beta}\right\},\\
N_{\alpha\beta} \doteq N_{\beta\alpha} &= -\left\{\frac{1}{AB}\left[\frac{\partial^2\Phi}{\partial\alpha\,\partial\beta} - \frac{1}{B}\frac{\partial B}{\partial\alpha}\frac{\partial\Phi}{\partial\beta} - \frac{1}{A}\frac{\partial A}{\partial\beta}\frac{\partial\Phi}{\partial\alpha}\right]\right\},\\
Q_\alpha &= -D\left\{\frac{1}{A}\frac{\partial}{\partial\alpha}(\nabla^2 w)\right\},\\
Q_\beta &= -D\left\{\frac{1}{B}\frac{\partial}{\partial\beta}(\nabla^2 w)\right\},\\
M_{\alpha\alpha} &= -D\left\{\frac{1}{A}\frac{\partial}{\partial\alpha}\left[\frac{1}{A}\frac{\partial w}{\partial\alpha}\right] + \frac{1}{AB^2}\frac{\partial A}{\partial\beta}\frac{\partial w}{\partial\beta} + \nu\left[\frac{1}{B}\frac{\partial}{\partial\beta}\left(\frac{1}{B}\frac{\partial w}{\partial\beta}\right) + \frac{1}{A^2B}\frac{\partial B}{\partial\alpha}\frac{\partial w}{\partial\alpha}\right]\right\},\\
M_{\beta\beta} &= -D\left\{\frac{1}{B}\frac{\partial}{\partial\beta}\left[\frac{1}{B}\frac{\partial w}{\partial\beta}\right] + \frac{1}{A^2B}\frac{\partial B}{\partial\alpha}\frac{\partial w}{\partial\alpha} + \nu\left[\frac{1}{A}\frac{\partial}{\partial\alpha}\left(\frac{1}{A}\frac{\partial w}{\partial\alpha}\right) + \frac{1}{AB^2}\frac{\partial A}{\partial\beta}\frac{\partial w}{\partial\beta}\right]\right\},\\
M_{\beta\alpha} \doteq M_{\alpha\beta} &= -D(1-\nu)\frac{1}{AB}\left\{\frac{\partial^2 w}{\partial\alpha\,\partial\beta} - \frac{1}{B}\frac{\partial B}{\partial\alpha}\frac{\partial w}{\partial\beta} - \frac{1}{A}\frac{\partial A}{\partial\beta}\frac{\partial w}{\partial\alpha}\right\}.
\end{aligned}
\tag{2}
$$

In order to be able to deal with a differential equation of the form

$$(\nabla^4 + m^2 \nabla_c^2) f = F,$$

it is convenient to introduce a complex dependent variable

$$V = w + i\,\omega\,\Phi.$$

Multiplying the first equation of (1) by $(i\,\omega)$ and adding it to the second equation yields

$$\nabla^4 (w + i\,\omega\,\Phi) - i\,\omega\,E\,h\,\nabla_c^2 \left(w - \frac{1}{i\,\omega\,E\,h\,D}\Phi\right) =$$

$$= \frac{p_n}{D} - \frac{\Gamma}{D}(c_\alpha + c_\beta) - i\,\omega\,\nabla^2 [(1-\nu)\,\Gamma + E\,h\,(\epsilon\,T)].$$

To secure a differential equation in $(w + i\,\omega\,\Phi)$ the condition

$$w - \frac{1}{i\,\omega\,E\,h\,D}\Phi = w + i\,\omega\,\Phi,$$

hence

$$\omega = \frac{\sqrt{12\,(1-\nu^2)}}{E\,h^2}.$$

Now equations (1) can be condensed into one complex differential equation

$$\{\nabla^4 - i\,E\,h\,\omega\,\nabla_c^2\}\,V = \frac{p_n}{D} - \Gamma\,(c_1 + c_2) - i\,\omega\,\nabla^2 [(1-\nu)\,\Gamma + E\,h\,(\epsilon\,T)]. \tag{2}$$

Original functions Φ and w can be expressed in terms of complex function V as follows

$$w = \tfrac{1}{2}(V + \bar{V}] \tag{a}$$

and

$$\Phi = -\frac{i}{2\,\omega}[V - \bar{V}], \tag{b}$$

where $\bar{V}$ is the conjugate of complex V.

For a shallow spherical shell

$$c_1 = c_2 = 1/R = \text{constant}$$

and the first fundamental quadratic form of its middle surface is expressed approximately by (fig. 1)

$$d\,s^2 = d\,r^2 + r^2\,d\,\theta^2 + d\,z^2 \approx d\,r^2 + r^2\,d\,\theta^2.$$

As $d\,z^2 \approx (r/R)^2\,d\,r^2$ and by hypothesis $r^2 \gg (r/R)^2 \ll 1$, then $|d\,z^2|$ is of small order of magnitude and can be suppressed.

For shallow spherical shells the coefficients of the fundamental metric are

$$A^2 = 1, \qquad B^2 = r^2$$

and the parametric coordinates

$$\alpha = r, \qquad \beta = \theta.$$

Inserting all these fundamental quantities of the shallow spherical shell into differential operators delivers

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$$

and

$$\nabla_c^2 = \frac{1}{R}\left[\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right] = \frac{1}{R}\nabla^2.$$

The sectional stress quantities can be expressed by

$$\left.\begin{aligned}
N_{rr} &= \left\{\frac{1}{r^2}\frac{\partial^2 \Phi}{\partial \theta^2} + \frac{1}{r}\frac{\partial \Phi}{\partial r}\right\} + \Gamma,\\
N_{\theta\theta} &= \frac{\partial^2 \Phi}{\partial r^2} + \Gamma,\\
N_{r\theta} &= N_{\theta r} \approx -\frac{\partial}{\partial r}\left[\frac{\partial \Phi}{r\,\partial \theta}\right],\\
Q_r &= -D\left\{\frac{\partial}{\partial r}(\nabla^2 w)\right\},\\
Q_\theta &= -D\left\{\frac{1}{r}\frac{\partial}{\partial \theta}(\nabla^2 w)\right\},\\
M_{rr} &= -D\left\{\frac{\partial^2 w}{\partial r^2} + \nu\left[\frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2} + \frac{1}{r}\frac{\partial w}{\partial r}\right]\right\},\\
M_{\theta\theta} &= -D\left\{\frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \nu\left(\frac{\partial^2 w}{\partial r^2}\right)\right\},\\
M_{r\theta} &\doteq -D(1-\nu)\left\{\frac{\partial}{\partial r}\left(\frac{\partial w}{r\,\partial \theta}\right)\right\},
\end{aligned}\right\} \tag{3}$$

where

$$p_\alpha = p_r, \qquad p_\beta = p_\theta.$$

Rabotnov [15] and Goldenveizer [8] have investigated the general shallow thin shell theory quite extensively for admissible simplifications.

Differential equation (2) simplifies to

$$\left\{\nabla^2\left[\nabla^2 - i\left(\frac{\sqrt{12(1-\nu^2)}}{Rh}\right)\right]\right\} V = \frac{p_n}{D} - \frac{2}{DR}\Gamma - i\,\omega\,\nabla^2[(1-\nu)\,\Gamma + E\,h\,(\epsilon\,T)]. \tag{4}$$

A somewhat shorter complex form of equation (4) was first set forth by Vekua [21]. The homogeneous part of equation (4) has also been given independently by E. Reissner [16].

Solution of equation (4) is expressible by a function

$$V = V_0 + V_1 + V_2, \tag{5}$$

where V_0 is a particular solution and the other arbitrary functions are given by equations

$$\nabla^2 V_1 = 0, \tag{6}$$

$$\{\nabla^2 - i\lambda^2\} V_2 = 0, \tag{7}$$

where
$$\lambda^2 = \frac{\sqrt{12(1-\nu^2)}}{R h}.$$

As pointed out by LOVE [11], solutions for equations of this type can be assumed in the form

$$V_1 = \sum_{n=0}^{\infty} v_n^1(r) \begin{Bmatrix}\cos n\theta \\ \sin n\theta\end{Bmatrix}$$

and

$$V_2 = \sum_{n=0}^{\infty} v_n^2(r) \begin{Bmatrix}\cos n\theta \\ \sin n\theta\end{Bmatrix}.$$

Substituting function V_1 in equation (6) yields a characteristic equation for v_n^1

$$r^2 \frac{d^2 v_n^1}{d r^2} + r \frac{d v_n^1}{d r} - n^2 v_n^1 = 0.$$

For $n=0$:
$$r \frac{d^2 v_n^1}{d r^2} + \frac{d v_n^1}{d r} = 0.$$

Then
$$v_n^1 = A_0 + B_0 Ln(r\lambda),$$

where A_0, B_0 are integration constants and $Ln(r\lambda)$ designates natural logarithm.

For $n=m \geqq 1$:
$$r^2 \frac{d^2 v_m^1}{d r^2} + r \frac{d v_m^1}{d r} - m^2 v_m^1 = 0.$$

This is an equidimensional equation and its solution is given by [10, 17]

$$v_m^1 = C_m r^m + D_m r^{-m},$$

where C_m and D_m are integration constants.

The complete solution for V_1 is then

$$V_1 = \begin{cases} A_0 + B_0 Ln(r\lambda) + \sum_{n=1}^{\infty} [C_n r^n + D_n \bar{r}^n] \cos n\theta, \\ \sum_{n=1}^{\infty} [\bar{C}_n r^n + \bar{D}_n \bar{r}^n] \sin n\theta, \end{cases} \tag{8}$$

where A_0, B_0, C_n, D_n, $\bar{C}_n$, $\bar{D}_n$ are integration constants.

Inserting the assumed function V_2 in (7) furnishes the characteristic equation for v_n^2

$$r^2 \frac{d^2 v_n^2}{d r^2} + r \frac{d v_n^2}{d r} - [i(r\lambda)^2 + n^2] v_n^2 = 0.$$

The solution of this equation is given in terms of modified Bessel functions [10]

$$v_n^2 = Z(i^{3/2} r\lambda) = \bar{S} J_n(i^{3/2} r\lambda) + \bar{T} Y_n(i^{3/2} r\lambda) = \hat{S}_n I_n(\sqrt{i}\, r\lambda) + \hat{T}_n K_n(\sqrt{i}\, r\lambda),$$

where $\bar{S}_n$, $\bar{T}_n$, $\hat{S}_n$ and $\hat{T}_n$ are integration constants. J_n, Y_n and I_n, K_n denote regular and modified Bessel functions respectively.

This solution in turn can be expressed by KELVIN functions [11] of n'th order, hence

$$v_n{}^2 = F_n [\mathrm{ber}_n (r\lambda) + i\,\mathrm{bei}_n (r\lambda)] + G_n [\mathrm{ker}_n (r\lambda) + i\,\mathrm{kei}_n (r\lambda)].$$

Complete solution for V_2 and now be set down

$$V_2 = \begin{cases} F_0 [\mathrm{ber}\,(r\lambda) + i\,\mathrm{bei}\,(r\lambda)] + G_0 [\mathrm{ker}\,(r\lambda) + i\,\mathrm{kei}\,(r\lambda)] + \\ + \sum\limits_{n=1}^{\infty} \{F_n [\mathrm{ber}_n (r\lambda) + i\,\mathrm{bei}_n (r\lambda)] + G_n [\mathrm{ker}_n (r\lambda) + i\,\mathrm{kei}_n (r\lambda)]\} \cos n\theta, \\ \sum\limits_{n=1}^{\infty} \{\bar{F}_n [\mathrm{ber}_n (r\lambda) + i\,\mathrm{bei}_n (r\lambda)] + \bar{G}_n [\mathrm{ker}_n (r\lambda) + i\,\mathrm{kei}_n (r\lambda)]\} \sin n\theta. \end{cases} \tag{9}$$

All arbitrary integration constants F_0, G_0, F_n, G_n, $\bar{F}_n$ and $\bar{G}_n$ contained in the solution of V are to be taken complex.

Particular Solution

Equations (1) for a spherical shell are

$$\begin{aligned} \nabla^4 \Phi - \frac{E h}{R} \nabla^2 w &= -(1-\nu) \nabla^2 \Gamma - E h \nabla^2 (\epsilon T), \\ \nabla^4 w + \frac{1}{D R} \nabla^2 \Phi &= \frac{p_n}{D} - \frac{2}{D R} \Gamma. \end{aligned} \tag{10}$$

The first equation of (10) can be rearranged in order to extract particular solutions, hence

$$\nabla^2 \left[\nabla^2 \Phi_p - \frac{E h}{R} w_p + (1-\nu) \Gamma + E h (\epsilon T)\right] = 0.$$

For symmetrical loading it follows

$$\nabla^2 \Phi_p = \frac{E h}{R} w_p - (1-\nu) \Gamma - E h (\epsilon T) + H + K\, Ln\, (r\lambda),$$

where H and K are integration constants.

Substituting this relation into the second equation of (10)

$$\nabla^4 w_p + \frac{E h}{D R^2} w_p = \frac{p_n}{D} - \frac{(1+\nu)}{D R} \Gamma + \frac{E h}{D R} (\epsilon T) - \frac{1}{D R} [H + K\, Ln\, (r\lambda)].$$

Then
$$w_p = \frac{R^2}{E h} p_n - \frac{(1+\nu) R}{E h} \Gamma + R (\epsilon T) - \frac{R}{E h} [H + K\, Ln\, (r\lambda)]$$

and
$$\nabla^2 \Phi_p = p_n R - 2\Gamma.$$

For antisymmetrical loading particular displacement function is given by

$$w_p = \frac{R^2}{E h} p_n - \frac{(1+\nu) R}{E h} \Gamma + R (\epsilon T) - \frac{R}{E h} \left[\bar{H} r + \frac{\bar{K}}{r}\right],$$

where $\bar{H}$ and $\bar{K}$ are integration constants.

Uniform Normal Load:

For uniform loading intensity

$$p_n = p_0,$$
$$\Gamma = 0.$$

Then the particular displacement function becomes

$$w_p = p_0 \frac{R^2}{E h} \tag{11}$$

and particular stress function is delivered by

$$\Phi_p = p_0 \frac{R r^2}{4}. \tag{12}$$

Wind Pressure:

A simplified wind loading can be presented by

$$p_n = p_w r \cos\theta / R,$$
$$p_\theta = p_r = \Gamma = 0.$$

For wind loading the particular displacement function is

$$w_p = p_w R \cos\theta / E h \tag{13}$$

and particular stress function is

$$\Phi_p = p_w r^3 \cos\theta / 8. \tag{14}$$

Parabolic Normal Load:

Loading function is given by

$$p_n = p_p r^2,$$
$$p_r = p_\theta = \Gamma = 0.$$

Particular displacement function is given by

$$w_p = p_p R^2 r^2 / E h \tag{15}$$

and particular stress function by

$$\Phi_p = p_p R r^4 / 16. \tag{16}$$

Uniform Horizontal Load:

Loading function can be taken in an approximate form

$$p_n = p_h \frac{r}{R} \cos\theta,$$
$$p_r = p_h \cos\theta,$$
$$p_\theta = -p_h \sin\theta,$$
$$\Gamma = -p_h r \cos\theta.$$

Particular displacement function is then

$$w_p (2+\nu) p_h r R \cos\theta / E h \tag{17}$$

and particular stress function

$$\Phi_p = 3 p_h r^3/_8. \tag{18}$$

Uniformly Symmetrical

Temperature Variation:

This condition results from a symmetrical temperature differential $T = T_0 =$ const. throughout the thin shell.

Displacement function is given by

$$w_p = R \epsilon T_0 \tag{19}$$

and stress function by

$$\Phi_p = 0. \tag{20}$$

Antimetrical Temperature

Variation:

The proposed temperature function $T = T_0 r \cos\theta / \bar{r}$ attempts to describe approximately the temperature variation in the skin of thin shell brought about by unilateral sunshine.

For such temperature variation the particular displacement function is

$$w_p = \epsilon T_0 R r \cos\theta / \bar{r} \tag{21}$$

and again the stress function

$$\Phi_p = 0. \tag{22}$$

General Solutions Applicable to Calotte Shells

In connection with all the problems considered in this paper the general solution is confined to the part of functions proportional to cosine.

Insertion of appropriate complex integration constants, particular solutions (11), (12) of uniform normal loading and symmetrical temperature variation (19), (20) into the general solutions (5) yields

$$\begin{aligned}
V &= V_0 + V_1 + V_2 = \\
&= \left\{ \frac{p_0 R^2}{E h} + R \epsilon T_0 + A_0{}' \operatorname{ber}(r\lambda) + A_0{}^2 \operatorname{bei}(r\lambda) + B_0{}' \ker(r\lambda) + B_0{}^2 \operatorname{kei}(r\lambda) + \right. \\
&\quad + F_0{}' Ln(r\lambda) + E_0{}' + \sum_{n=1}^{\infty} [A_n{}' \operatorname{ber}_n(r\lambda) + A_n{}^2 \operatorname{bei}_n(r\lambda) + B_n{}' \ker_n(r\lambda) + \\
&\quad \left. + B_n{}^2 \operatorname{kei}_n(r\lambda) + C_n{}' r^n + D_n{}' r^{-n}] \cos n\theta \right\} + \\
&\quad + i \left\{ \frac{\sqrt{12(1-\nu^2)}}{4 E h} p_0 R r^2 + A_0{}' \operatorname{bei}(r\lambda) - A_0{}^2 \operatorname{ber}(r\lambda) + B_0{}' \operatorname{kei}(r\lambda) - B_0{}^2 \ker(r\lambda) + \right. \\
&\quad + F_0{}^2 Ln(r\lambda) + E_0{}^2 + \sum_{n=1}^{\infty} [A_n{}' \operatorname{bei}_n(r\lambda) - A_n{}^2 \operatorname{ber}_n(r\lambda) + B_n{}' \operatorname{kei}_n(r\lambda) - \\
&\quad \left. - B_n{}^2 \ker_n(r\lambda) + C_n{}^2 r^n + D_n{}^2 r^{-n}] \cos n\theta \right\}.
\end{aligned}$$

Integration constants $A_n{}^1$, $A_n{}^2$, $B_n{}^1$, $B_n{}^2$, $C_n{}^1$, $C_n{}^2$, $F_0{}^1$, $F_0{}^2$, $E_0{}^1$ and $E_0{}^2$ $(n = 0, 1, 2 \ldots)$ are all real.

The final solutions of basic parametric functions Φ, w can be obtained from this relation by imposition of equations (a) and (b).

Imposing (a) yields displacement function

$$w = \frac{p_0 R^2}{E h} + R \epsilon T_0 + A_0' \operatorname{ber}(r\lambda) + A_0^2 \operatorname{bei}(r\lambda) + B_0' \operatorname{ker}(r\lambda) + B_0^2 \operatorname{kei}(r\lambda) + \\ + F_0' Ln(r\lambda) + E_0' + \sum_{n=1}^{\infty} [A_n' \operatorname{ber}_n(r\lambda) + A_n^2 \operatorname{bei}_n(r\lambda) + B_n' \operatorname{ker}_n(r\lambda) + \\ + B_n^2 \operatorname{kei}_n(r\lambda) + C_n' r^n + D_n' r^{-n}] \cos n\theta. \tag{23}$$

Stress function is delivered by (b)

$$\Phi = p_0 \frac{R r^2}{4} + \frac{E h}{\sqrt{12(1-\nu^2)}} \{A_0' \operatorname{bei}(r\lambda) - A_0^2 \operatorname{ber}(r\lambda) + B_0' \operatorname{kei}(r\lambda) - B_0^2 \operatorname{ker}(r\lambda) + \\ + F_0^2 Ln(r\lambda) + E_0^2 + \sum_{n=1}^{\infty} [A_n' \operatorname{bei}_n(r\lambda) - A_n^2 \operatorname{ber}_n(r\lambda) + B_n' \operatorname{kei}_n(r\lambda) - \\ - B_n^2 \operatorname{ker}_n(r\lambda) + C_n^2 r^n + D_n^2 r^{-n}] \cos n\theta\}. \tag{24}$$

If these functions are going to depict the state of stress and deformation of calotte shells, it is necessary to let proper boundary conditions of the shell determine all the integration constants contained in solutions (23) and (24).

In order that this operation can be carried out, individual types of shells have to be taken under consideration.

Boundary Conditions Relevant to Various Types of Spherical Calotte Shells

Spherical Shells Over Polygonal Type of Base

A few shells of this type are illustrated in fig. 3.

In general these shells exhibit two boundaries at $r = r_i$ and $r = \bar{r}$ (fig. 4).

Assuming that the inside boundary is limited to small concentric hole $(r = r_i, r_i/\bar{r} \ll 1)$ or to a small heavily loaded circular area $(r = r_p, r_p/\bar{r} \ll 1)$ then the stress distribution around this location is practically rotationally symmetrical and the stress function and the normal displacement function can be given approximately by

$$\Phi = p_0 \frac{R r^2}{4} + \frac{E h}{\sqrt{12(1-\nu^2)}} \{B_0' \operatorname{kei}(r\lambda) - B_0^2 \operatorname{ker}(r\lambda) + F_0^2 Ln(r\lambda)\},$$

$$w = \frac{p_0 R^2}{E h} + R(\epsilon T_0) + B_0' \operatorname{ker}(r\lambda) + B_0^2 \operatorname{kei}(r\lambda).$$

This uncoupling of boundary effects is possible for thin shells, which dampen out transverse bending perturbations in a very short distance from the origin of disturbance. In other words the inside boundary disturbance does not affect the stress nor strain of the outside boundary and vice versa.

The solution for the case of a concentrated load over a small area at the apex of the shell has been given by E. Reissner [16].

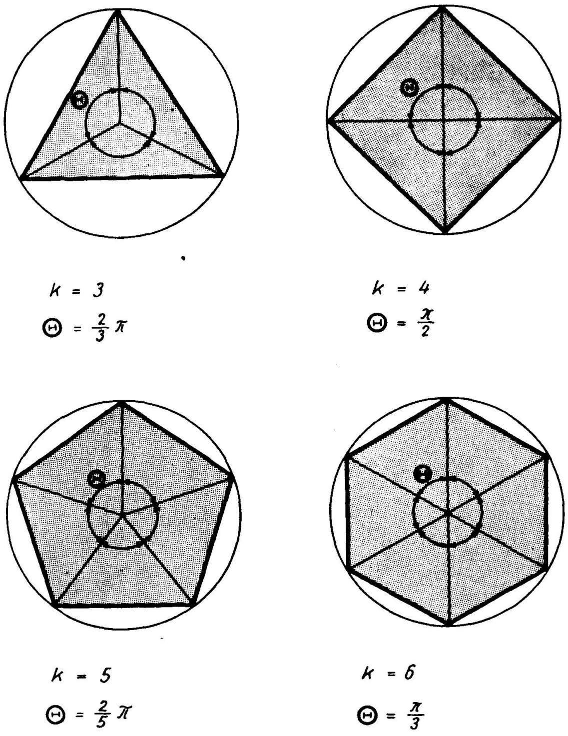

Fig. 3. Thin Shells of Polygonal Type with k Sides.

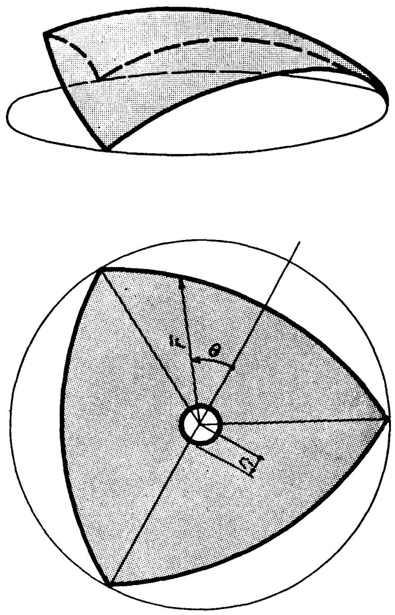

Fig. 4. Calotte Shell of Triangular Type Exhibiting Boundary Coordinates.

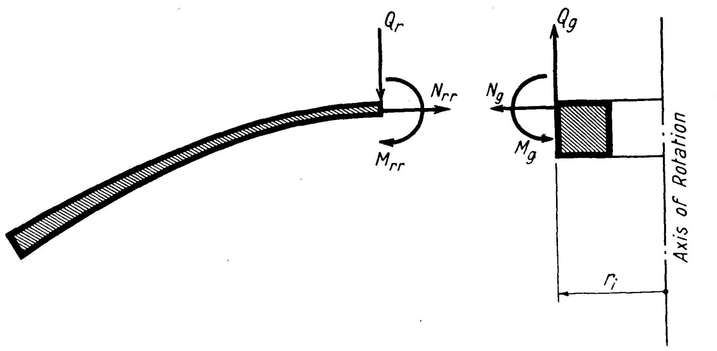

Fig. 5. Exploded Section at Apex Opening of Shell.

In case of a central circular opening that is lined with a circular girder, the proper boundary conditions are (fig. 5).

At r_i:

$$\begin{aligned} N_{rr} &= N_g, \\ Q_{rr} &= Q_g, \\ M_{rr} &= M_g, \\ \epsilon_\theta &= \epsilon_g, \\ \underset{\text{shell}}{[dw/dr]} &= [dw/dr]_g, \end{aligned}$$

where ϵ_θ and ϵ_g designate circumferential strain of the shell and ring girder respectively.

That part of the solution, which depends largely upon the influence of the exterior boundary of the shell, is

$$w = \frac{p_0 R^2}{E h} + R(\epsilon T_0) + A_0' \operatorname{ber}(r\lambda) + A_0^2 \operatorname{bei}(r\lambda) + E_0' + \sum_{n=1}^{\infty} [A_n' \operatorname{ber}_n(r\lambda) + A_n^2 \operatorname{bei}_n(r\lambda) + C_n' r^n] \cos n\theta$$

and

$$\Phi = p_0 \frac{R r^2}{4} + \frac{E h^2}{\sqrt{12(1-\nu^2)}} \{A_0' \operatorname{bei}(r\lambda) - A_0^2 \operatorname{ber}(r\lambda) + E_0^2 + \sum_{n=1}^{\infty} [A_n' \operatorname{bei}_n(r\lambda) - A_n^2 \operatorname{ber}_n(r\lambda) + C_n^2 r^n] \cos n\theta\}.$$

If the shallow spherical shell encloses a base of regular polygon with k — sides, then its state of stress and strain also acquires a k — fold pattern of polar symmetry about the apex of the shell. Accordingly thin calotte shell's normal displacement function and stress function exhibit the same k — fold characteristics, hence

$$w = p_0 \frac{R^2}{E h} + R \epsilon T_0 + A_0^2 \operatorname{bei}(r\lambda) + A_0' \operatorname{ber}(r\lambda) + E_0' + \sum_{n=1}^{\infty} [A_{kn}^2 \operatorname{bei}_{kn}(r\lambda) + A_{kn}' \operatorname{ber}_{kn}(r\lambda) + C_{kn}' r^{kn}] \cos k n\theta, \tag{25}$$

$$\Phi = p_0 \frac{R r^2}{4} + \frac{E h^2}{\sqrt{12(1-\nu^2)}} \{A_0' \operatorname{bei}(r\lambda) - A_0^2 \operatorname{ber}(r\lambda) + E_0^2 + \sum_{n=1}^{\infty} [A_{kn}' \operatorname{bei}_{kn}(r\lambda) - A_{kn}^2 \operatorname{ber}_{kn}(r\lambda) + C_{kn}^2 r^{kn}] \cos k n\theta\}. \tag{26}$$

Satisfaction of Edge Conditions by Boundary Collocation

An exact solution of the edge restraints for polygonal shells is for practical purpose quite impossible, therefore it is advisable to resort to the method of collocation in order to secure at least a point by point satisfaction of the prescribed approximate boundary conditions.

For calotte shells the following boundary conditions seem best suited to describe a reasonable edge behavior.

Hence at $r = \bar{r}$:

$$N_{nn} = 0, \tag{c}$$
$$M_{nn} = 0, \tag{d}$$
$$\epsilon_t = 0, \tag{e}$$
$$w = 0, \tag{f}$$

where $\bar{r}$ radius to collocation points on shell's boundary,
ϵ_t strain tangential to the boundary of the shell.

For a central symmetry point, like points (3) in fig. 8, boundary condition (f) satisfies also condition (e) approximately, hence

$$N_{nn} = 0, \tag{c}$$

$$M_{nn} = 0, \tag{d}$$

$$w = 0. \tag{f}$$

For many shells encountered in practice the boundary condition (d) could be replaced by

$$\frac{\partial w}{\partial n} = 0 \tag{g}$$

or by

$$M_{nn} = M,$$

where M is an applied moment distributed along the outside edge.

Boundary condition (c) can be expressed by stress resultants in polar coordinates (fig. 6)

$$N_{nn} = N_{rr} \cos^2 \xi + N_{\theta\theta} \sin^2 \xi + 2\, N_{r\theta} \sin \xi \cos \xi = 0$$

or

$$\left[\frac{1}{r^2}\frac{\partial^2 \Phi}{\partial \theta^2} + \frac{1}{r}\frac{\partial \Phi}{\partial r}\right] \cos^2 \xi + \left[\frac{\partial^2 \Phi}{\partial r^2}\right] \sin^2 \xi - \left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \Phi}{\partial \theta}\right)\right] \sin 2\xi = 0.$$

The angle formed by the radius vector and boundary normal is measured by ξ.

Relation (d) can be expressed by stress couples in polar coordinates (fig. 7).

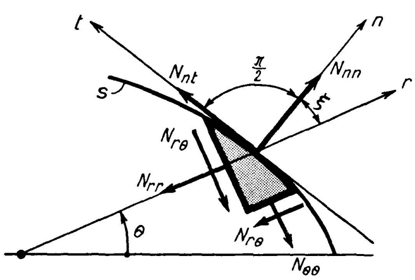

Fig. 6. Stress Resultants Acting at Boundary of Shell.

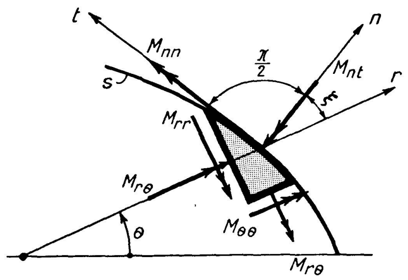

Fig. 7. Stress Couples Acting at the Boundary of Shell.

$$M_{nn} = M_{rr} \cos^2 \xi + M_{\theta\theta} \sin^2 \xi + 2\, M_{r\theta} \sin \xi \cos \xi = 0$$

or

$$\left[\frac{\partial^2 w}{\partial r^2} + \nu \left(\frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}\right)\right] \cos^2 \xi +$$

$$+ \left[\frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2} + \nu \left(\frac{\partial^2 w}{\partial r^2}\right)\right] \sin^2 \xi - (1-\nu) \left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial w}{\partial \theta}\right)\right] \sin 2\xi = 0.$$

Boundary condition (e) is

$$\epsilon_t = \epsilon_r \cos^2\left(\xi + \frac{\pi}{2}\right) + \epsilon_\theta \sin^2\left(\xi + \frac{\pi}{2}\right) + \gamma_{r\theta} \sin\left(\xi + \frac{\pi}{2}\right) \cos\left(\xi + \frac{\pi}{2}\right) =$$

$$= \epsilon_r \sin^2 \xi + \epsilon_\theta \cos^2 \xi - \gamma_{r\theta} \sin \xi \cos \xi = 0$$

or $$E\,h\,\epsilon_t = [N_{rr} - \nu\,N_{\theta\theta}]\sin^2\xi + [N_{\theta\theta} - \nu\,N_{rr}]\cos^2\xi - (1+\nu)\,N_{r\theta}\sin 2\xi = 0,$$

where

$$\epsilon_r = \frac{\partial u}{\partial r} + \frac{w}{R}, \qquad \epsilon_\theta = \frac{1}{r}\frac{\partial v}{\partial\theta} + \frac{u}{r} + \frac{w}{R}, \qquad \gamma_{r\theta} = \frac{1}{r}\frac{\partial u}{\partial\theta} + r\frac{\partial}{\partial r}\left(\frac{v}{r}\right),$$

$$E\,h\,\epsilon_r = N_{rr} - \nu\,N_{\theta\theta}, \qquad E\,h\,\epsilon_\theta = N_{\theta\theta} - \nu\,N_{rr}, \qquad E\,h\,\gamma_{r\theta} = 2\,(1-\nu)\,N_{r\theta}$$

and u, v, w are meridional, circumferential and normal components of displacement (fig. 1).

Inserting stress function gives

$$E\,h\,\epsilon_t = \left[\frac{1}{r}\frac{\partial\Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\Phi}{\partial\theta^2} - \nu\frac{\partial^2\Phi}{\partial r^2}\right]\sin^2\xi + \left[\frac{\partial^2\Phi}{\partial r^2} - \nu\left(\frac{1}{r}\frac{\partial\Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\Phi}{\partial\theta^2}\right)\right]\cos^2\xi +$$
$$+ (1+\nu)\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\Phi}{\partial\theta}\right)\right]\sin 2\xi = 0.$$

Boundary condition (g) is

$$\frac{\partial w}{\partial n} = \left(\frac{\partial w}{\partial r}\right)\cos\xi + \left(\frac{1}{r}\frac{\partial w}{\partial\theta}\right)\sin\xi = 0.$$

Boundary conditions (c), (d), (e) and (f) are:

$$A_0{}^1\psi_1 + A_0{}^2\psi_2 + \sum_{n=1}^{\infty}[A_{kn}^1\psi_3 + A_{kn}^2\psi_4 + C_{kn}^2\psi_5\,] = -\frac{p_0\,R}{2}, \tag{c*}$$

$$A_0{}^1\psi_6 + A_0{}^2\psi_7 + \sum_{n=1}^{\infty}[A_{kn}^1\psi_8 + A_{kn}^2\psi_9 + C_{kn}^1\psi_{10}] = 0, \tag{d*}$$

$$A_0{}^1\psi_{11} + A_0{}^2\psi_{12} + \sum_{n=1}^{\infty}[A_{kn}^1\psi_{13} + A_{kn}^2\psi_{14} + C_{kn}^2\psi_{12}] = -(1-\nu)\frac{p_0\,R}{2}, \tag{e*}$$

$$E_0{}^1 + A_0{}^1\operatorname{ber}\mu + A_0{}^2\operatorname{bei}\mu + \sum_{n=1}^{\infty}[A_{kn}^1\operatorname{ber}_{kn}\mu + A_{kn}^2\operatorname{bei}_{kn}\mu + C_{kn}^1\bar{r}^{kn}]\cos n\,\bar\theta =$$
$$= -\left[\frac{p_0\,R^2}{E\,h} + R\,(\epsilon\,T_0)\right]. \tag{f*}$$

Boundary conditions (g) yield:

$$A_0{}^1\psi_{16} + A_0{}^2\psi_{17} + \sum_{n=1}^{\infty}[A_{kn}^1\psi_{18} + A_{kn}^2\psi_{19} + C_{kn}^1\psi_{20}] = 0. \tag{g*}$$

Coefficients in the boundary relations are given by:

$$\psi_1 = \left[\frac{1}{\omega}\left(\frac{\lambda}{\bar r}\right)\operatorname{bei}'\mu\right]\cos^2\bar\xi + \left[\frac{\lambda^2}{\omega}\operatorname{bei}''\mu\right]\sin^2\bar\xi,$$

$$\psi_2 = -\left[\frac{1}{\omega}\left(\frac{\lambda}{\bar r}\right)\operatorname{ber}'\mu\right]\cos^2\bar\xi - \left[\frac{\lambda^2}{\omega}\operatorname{ber}''\mu\right]\sin^2\bar\xi,$$

$$\psi_3 = \left\{\left[-\frac{1}{\omega}\left(\frac{k\,n}{\bar r}\right)^2\operatorname{bei}_{kn}\mu + \frac{1}{\omega}\left(\frac{\lambda}{\bar r}\right)\operatorname{bei}_{kn}'\mu\right]\cos^2\bar\xi + \left[\frac{\lambda^2}{\omega}\operatorname{bei}_{kn}''\mu\right]\sin^2\bar\xi\right\}\cos k\,n\,\bar\theta +$$
$$+ \left\{\left[\frac{k\,n}{\omega}\left(\frac{\lambda}{\bar r}\right)\operatorname{bei}_{kn}'\mu - \frac{k\,n}{\omega}\frac{1}{\bar r^2}\operatorname{bei}_{kn}\mu\right]\sin 2\bar\xi\right\}\sin k\,n\,\bar\theta,$$

$$\psi_4 = \left\{\left[\frac{1}{\omega}\left(\frac{k\,n}{\bar{r}}\right)^2 \mathrm{ber}_{kn}\,\mu - \frac{1}{\omega}\left(\frac{\lambda}{\bar{r}}\right)\mathrm{ber}'_{kn}\,\mu\right]\cos^2\bar{\xi} + \left[\frac{\lambda^2}{\omega}\mathrm{ber}''_{kn}\,\mu\right]\sin^2\bar{\xi}\right\}\cos k\,n\,\bar{\theta} +$$
$$+ \left\{\left[-\frac{\lambda}{\omega}\left(\frac{k\,n}{\bar{r}}\right)\mathrm{ber}'_{kn}\,\mu + \frac{k\,n}{\omega}\,\frac{1}{\bar{r}^2}\mathrm{ber}_{kn}\,\mu\right]\sin 2\bar{\xi}\right\}\sin k\,n\,\bar{\theta},$$

$$\psi_5 = \bar{r}^{kn-2}\frac{k\,n}{\omega}(k\,n-1)\left[(1-2\cos^2\bar{\xi})\cos k\,n\,\bar{\theta} + \sin 2\bar{\xi}\sin k\,n\,\bar{\theta}\right],$$

$$\psi_6 = -\left[\nu\left(\frac{\lambda}{\bar{r}}\right)\mathrm{ber}'\,\mu + \lambda^2\,\mathrm{ber}''\,\mu\right]\cos^2\bar{\xi} - \left[\frac{\lambda}{\bar{r}}\mathrm{ber}'\,\mu + \nu\,\lambda^2\,\mathrm{ber}''\,\mu\right]\sin^2\bar{\xi},$$

$$\psi_7 = -\left[\nu\left(\frac{\lambda}{\bar{r}}\right)\mathrm{bei}'\,\mu + \lambda^2\,\mathrm{bei}''\,\mu\right]\cos^2\bar{\xi} - \left[\left(\frac{\lambda}{\bar{r}}\right)\mathrm{bei}'\,\mu + \nu\,\lambda^2\,\mathrm{bei}''\,\mu\right]\sin^2\bar{\xi},$$

$$\psi_8 = -\left\{\left[\nu\left(\frac{\lambda}{\bar{r}}\right)\mathrm{ber}'_{kn}\,\mu - \nu\left(\frac{k\,n}{\bar{r}}\right)^2\mathrm{ber}_{kn}\,\mu + \lambda^2\,\mathrm{ber}''_{kn}\,\mu\right]\cos^2\bar{\xi} +\right.$$
$$+ \left.\left[\left(\frac{\lambda}{\bar{r}}\right)\mathrm{ber}'_{kn}\,\mu - \left(\frac{k\,n}{\bar{r}}\right)^2\mathrm{ber}_{kn}\,\mu + \nu\,\lambda^2\,\mathrm{ber}''_{kn}\,\mu\right]\sin^2\bar{\xi}\right\}\cos k\,n\,\bar{\theta} +$$
$$+ \left\{(1-\nu)\,k\,n\left[\left(\frac{\lambda}{\bar{r}}\right)\mathrm{ber}'_{kn}\,\mu - \frac{1}{\bar{r}^2}\mathrm{ber}_{kn}\,\mu\right]\sin 2\bar{\xi}\right\}\sin k\,n\,\bar{\theta},$$

$$\psi_9 = -\left\{\left[\nu\left(\frac{\lambda}{\bar{r}}\right)\mathrm{bei}'_{kn}\,\mu - \nu\left(\frac{k\,n}{\bar{r}}\right)^2\mathrm{bei}_{kn}\,\mu + \lambda^2\,\mathrm{bei}''_{kn}\,\mu\right]\cos^2\bar{\xi} +\right.$$
$$+ \left.\left[\left(\frac{\lambda}{\bar{r}}\right)\mathrm{bei}'_{kn}\,\mu - \left(\frac{k\,n}{\bar{r}}\right)^2\mathrm{bei}_{kn}\,\mu + \nu\,\lambda^2\,\mathrm{bei}''_{kn}\,\mu\right]\sin^2\bar{\xi}\right\}\cos k\,n\,\bar{\theta} +$$
$$+ \left\{(1-\nu)\,k\,n\left[\left(\frac{\lambda}{\bar{r}}\right)\mathrm{bei}'_{kn}\,\mu - \frac{1}{\bar{r}^2}\mathrm{bei}_{kn}\,\mu\right]\sin 2\bar{\xi}\right\}\sin k\,n\,\bar{\theta},$$

$$\psi_{10} = (1-\nu)\,k\,n\,(k\,n-1)\,\bar{r}^{kn-2}\left[(1-2\cos\xi^2)\cos k\,n\,\bar{\theta} + \sin 2\bar{\xi}\sin k\,n\,\bar{\theta}\right],$$

$$\psi_{11} = \left[\frac{1}{\omega}\left(\frac{\lambda}{\bar{r}}\right)\mathrm{bei}'\,\mu - \nu\frac{\lambda}{\omega}\mathrm{bei}''\,\mu\right]\sin^2\bar{\xi} + \left[\frac{\lambda^2}{\omega}\mathrm{bei}''\,\mu - \frac{\nu}{\omega}\left(\frac{\lambda}{\bar{r}}\right)\mathrm{bei}'\,\mu\right]\cos^2\bar{\xi},$$

$$\psi_{12} = \left[\nu\frac{\lambda^2}{\omega}\mathrm{ber}''\,\mu - \frac{1}{\omega}\left(\frac{\lambda}{\bar{r}}\right)\mathrm{ber}'\,\mu\right]\sin^2\bar{\xi} + \left[\frac{\nu}{\omega}\left(\frac{\lambda}{\bar{r}}\right)\mathrm{ber}'\,\mu - \frac{\lambda^2}{\omega}\mathrm{ber}''\,\mu\right]\cos^2\bar{\xi},$$

$$\psi_{13} = \left\{\left[-\frac{1}{\omega}\left(\frac{k\,n}{\bar{r}}\right)^2\mathrm{bei}_{kn}\,\mu + \frac{1}{\omega}\left(\frac{\lambda}{\bar{r}}\right)\mathrm{bei}'_{kn}\,\mu - \nu\frac{\lambda^2}{\omega}\mathrm{bei}''_{kn}\,\mu\right]\sin^2\xi +\right.$$
$$+ \left.\left[\frac{\lambda^2}{\omega}\mathrm{bei}''_{kn}\,\mu + \frac{\nu}{\omega}\left(\frac{k\,n}{\bar{r}}\right)^2\mathrm{bei}_{kn}\,\mu - \frac{\nu}{\omega}\left(\frac{\lambda}{\bar{r}}\right)\mathrm{bei}_{kn}\,\mu\right]\cos^2\bar{\xi}\right\}\cos k\,n\,\bar{\theta} -$$
$$- \left\{\frac{(1+\nu)}{\omega}k\,n\left[\left(\frac{\lambda}{\bar{r}}\right)\mathrm{bei}'_{kn}\,\mu - \frac{1}{\bar{r}^2}\mathrm{bei}_{kn}\,\mu\right]\sin 2\bar{\xi}\right\}\sin k\,n\,\bar{\theta},$$

$$\psi_{14} = \left\{\left[\frac{1}{\omega}\left(\frac{k\,n}{\bar{r}}\right)^2\mathrm{ber}_{kn}\,\mu - \frac{1}{\omega}\left(\frac{\lambda}{\bar{r}}\right)\mathrm{ber}'_{kn}\,\mu - \nu\frac{\lambda^2}{\omega}\mathrm{ber}''_{kn}\,\mu\right]\sin^2\bar{\xi} +\right.$$
$$+ \left.\left[\frac{\nu}{\omega}\left(\frac{\lambda}{\bar{r}}\right)\mathrm{ber}'_{kn}\,\mu - \frac{\nu}{\omega}\left(\frac{k\,n}{\bar{r}}\right)^2\mathrm{ber}_{kn}\,\mu + \frac{\lambda^2}{\omega}\mathrm{ber}''_{kn}\,\mu\right]\cos^2\bar{\xi}\right\}\cos k\,n\,\bar{\theta} +$$
$$+ \left\{(1+\nu)\frac{k\,n}{\omega}\left[\left(\frac{\lambda}{\bar{r}}\right)\mathrm{ber}'_{kn}\,\mu - \frac{1}{\bar{r}^2}\mathrm{ber}_{kn}\,\mu\right]\sin 2\bar{\xi}\right\}\sin k\,n\,\bar{\theta},$$

$$\psi_{15} = \frac{(1+\nu)}{\omega}k\,n\,(k\,n-1)\,\bar{r}^{kn-2}\left[(2\cos^2\xi - 1)\cos k\,n\,\bar{\theta} - \sin 2\bar{\xi}\sin k\,n\,\bar{\theta}\right],$$

$$\psi_{16} = [\lambda \operatorname{ber}' \mu] \cos \bar{\xi},$$
$$\psi_{17} = [\lambda \operatorname{bei} \mu) \cos \bar{\xi},$$
$$\psi_{18} = \{(\lambda \operatorname{ber}'_{kn} \mu) \cos \bar{\xi}\} \cos k n \bar{\theta} - \left\{\left[\left(\frac{k n}{\bar{r}}\right) \operatorname{ber}_{kn} \mu\right] \sin \bar{\xi}\right\} \sin k n \bar{\theta},$$
$$\psi_{19} = \{(\lambda \operatorname{bei}'_{kn}) \cos \bar{\xi}\} \cos k n \bar{\theta} - \left\{\left[\left(\frac{k n}{\bar{r}}\right) \operatorname{bei}_{kn} \mu\right] \sin \bar{\xi}\right\} \sin k n \bar{\theta},$$
$$\psi_{20} = \{[k n \bar{r}^{kn-1}] \cos \bar{\xi}\} \cos k n \bar{\theta} - \{[k n \bar{r}^{kn-1}] \sin \bar{\xi}\} \sin k n \bar{\theta},$$

with $(\mu = \bar{r}\lambda)\, \bar{r}, \bar{\theta}, \bar{\xi}$-coordinates of collocation points and where $\operatorname{ber}'_{kn} \mu$, $\operatorname{ber}''_{kn} \mu$ indicate first and second derivatives of Kelvin functions of $(k n)'$th order with respect to μ. Second derivatives of Kelvin functions can be expressed by

$$\operatorname{ber}''_{kn} \mu = -\frac{1}{\mu} \operatorname{ber}'_{kn} \mu + \left(\frac{k n}{\mu}\right)^2 \operatorname{ber}_{kn} \mu - \operatorname{bei}_{kn} \mu,$$
$$\operatorname{bei}''_{kn} \mu = -\frac{1}{\mu} \operatorname{bei}_{kn} \mu + \left(\frac{k n}{\mu}\right)^2 \operatorname{bei}_{kn} \mu + \operatorname{ber}_{kn} \mu$$

and the third derivatives of Kelvin functions by

$$\operatorname{ber}'''_{kn} \mu = \frac{(k n)^2 + 2}{\mu^2} \operatorname{ber}'_{kn} \mu - \operatorname{bei}'_{kn} \mu - 2 \frac{(k n)^2}{\mu^3} \operatorname{ber}_{kn} \mu + \frac{1}{\mu} \operatorname{bei}_{kn} \mu,$$
$$\operatorname{bei}'''_{kn} \mu = \frac{(k n)^2 + 2}{\mu^2} \operatorname{bei}'_{kn} \mu + \operatorname{ber}'_{kn} \mu - 2 \frac{(k n)^2}{\mu^3} \operatorname{bei}_{kn} \mu - \frac{1}{\mu} \operatorname{ber}_{kn} \mu.$$

In order to determine the number of terms to be used in any particular solution, the pattern of collocation points on the shell's boundary has to be established. Naturally an increase in the number of boundary points, where all prescribed boundary conditions are satisfied, increases the accuracy of the ultimate solution.

Fig. 8 illustrates that the same number of collocation points on the boundary of a hexagonal shell yields a more accurate solution than if they were located on the boundary of a triangular type of shell. Yet the computational effort spent to obtain the solution is identical.

In order to make the accuracy of the triangular type of shell comparable to that of the hexagonal shell, the number of its collocation points have to be increased. This adds considerably to the amount of work required to carry through the calculation. For instance, inclusion of one more collocation point requires additional four terms in the solution series. Obviously polygonal

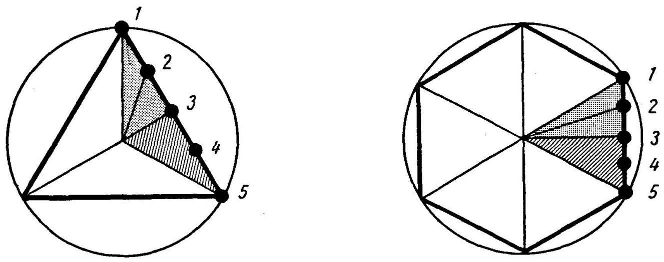

Fig. 8. Collocation Pattern on the Boundary of Polygonal Shell.

shells, with larger number of sides, require smaller number of collocation points to yield satisfactory solution and, therefore, need fewer terms in its series solutions.

It should be noted that due to k-ply symmetry of the problem only collocation points 1, 2 and 3 have to be satisfied for boundary conditions. Collocation points 4 and 5 are identically satisfied due to the symmetry of solution functions.

In general, the number of coefficients required in the stress and displacement functions Φ and w depend upon the number of collocation points assigned to the problem. For instance, in case of a triangular base, with five collocation points per side, it is necessary to take $n=1,2$ in series (25) and (26). For a calotte shell over a square base with five collocation points per side, the solution is

$$w = \frac{p_0 R^2}{E h} + R\epsilon T_0 + A_0{}^1 \operatorname{ber}(r\lambda) + A_0{}^2 \operatorname{bei}(r\lambda) + E_0{}^1 + \sum_{n=1}^{2} [A_{4n}^1 \operatorname{ber}_{4n}(r\lambda) + A_{4n}^2 \operatorname{bei}_{4n}(r\lambda) + C_{4n}^1 r^{4n}] \cos 4n\theta$$

and

$$\Phi = \frac{p_0 R r^2}{4} + \frac{E h^2}{\sqrt{12(1-\nu^2)}} \{A_0{}^1 \operatorname{bei}(r\lambda) - A_0{}^2 \operatorname{ber}(r\lambda) + \sum_{n=1}^{2} [A_{4n}^1 \operatorname{bei}_{4n}(r\lambda) - A_{4n}^2 \operatorname{ber}_{4n}(r\lambda) + C_{4n}^2 r^{4n}] \cos 4n\theta\}.$$

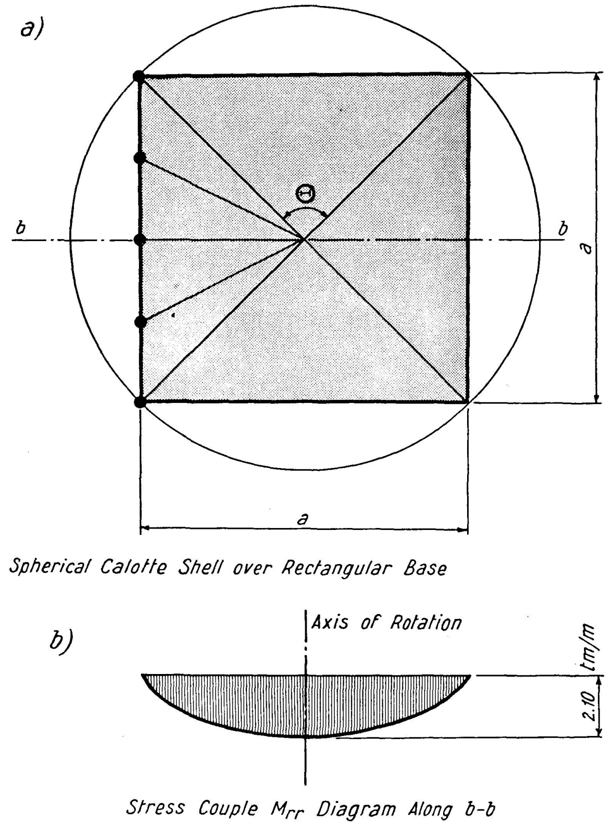

Fig. 9. Square Calotte Shell and Stress Couple Diagram under Uniform Normal Loading.

An extremely shallow shell over a square base with the following dimensions and properties (fig. 9a)

$$R = 100\,\text{m}, \quad h = 0{,}10\,\text{m}, \quad a = 11{,}5\,\text{m}, \quad \nu = 0{,}17, \quad \lambda = 0{,}59 \quad \Theta = \frac{\pi}{2}$$

is subjected to a uniform normal loading intensity of $p = 0{,}5\ \text{t/m}^2$. Then after some very extensive calculations, the stress couple M_{rr} variation along section $b-b$, shown in fig. 9b, is procured (see Appendix).

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Bibliography

1. A. Aas Jakobsen, «Kugelschalen über vier- und vieleckigem Grundriß», Publ. Int., Asso. Bridge and Struct. Engng., v. 5, 1937/38, p. 1.
2. — «Beitrag zur Theorie der Kugelschale auf Einzelstützen», Ing.-Arch., v. 8, 1937, p. 275.
3. F. R. Berman, "Analysis of Shallow Domes", D. Sc. Thesis, Mass. Inst. Tech., 1946.
4. J. C. Berry, "On Thin Hemispherical Shells Subjected to Concentrated Edge Moments and Forces", Proc. 2nd Midwestern Conf. Solid Mech., Purdue Univ., Sept. 8—9, 1955, Research Bull. No. 129, p. 25.
5. P. Csonka, "Calculation of Calotte Shells over Rectangular Bases", Acta Technica, Acad. Scien. Hung., v. 11, No. 3/4, 1955, p. 427.
6. — "Calotte Shell over Rectangular Base", Acta Technica, Acad. Scien. Hung., v. 13, No. 1/2, 1955, p. 150.
7. F. Dischinger, «Die Rotationsschalen mit unsymmetrischer Form und Belastung», Der Bauingenieur, v. 16, Nos. 35—38, 1935.
8. A. L. Goldenveizer, "Nekotorye priemy integrirovaniya uravnenii teorii tonkikh obolochek" (Several Methods of Integration for the Equations of the Theory of Thin Shells), Prikl. Mat. i Mekh., v. 10, No. 3, 1946, p. 3.
9. A. Havers, «Asymptotische Biegetheorie der unbelasteten Kugelschalen», Ing.-Arch. v. 6, 1935, p. 282.
10. H. B. Hildebrand, "Advanced Calculus for Engineers", Prentice-Hall, New York, 1949.
11. A. E. H. Love, "A Treatise on the Mathematical Theory of Elasticity", Dover, New York, 1944.
12. A. H. Mushatari, "Nekotorye obobshcheriya teorii tonkikh obolochek" (Several Generalizations on the Theory of Thin Shells), Izvestiya Fiziko-Matematicheskovo obshchestva pri Kazanskom Universitete, v. 11, Series 8, 1938.
13. A. Pucher, «Die Berechnung der Dehnungsspannungen von Rotationsschalen mit Hilfe von Spannungsfunktionen», Publ. Int. Asso. Bridge and Struct. Engng., v. 5, 1937/38, p. 275.

14. — «Über die Spannungsfunktion beliebig gekrümmter dünner Schalen», Proc. V Int. Cong. Appl. Mech. (Cambridge, Mass.), 1938, p. 134.
15. Y. N. RABOTNOV, "The Equations for the Boundary Zone in the Theory of Shells", Doklady, Akad. Nauk U.S.S.R., v. 17, No. 5, 1945, p. 329.
16. E. REISSNER, "Stresses and Small Displacements of Shallow Spherical Shells", Jour. Math. and Phys., v. 25, No. 1, 1946, p. 80.
17. E. REISSNER and W. T. MARTIN, "Elementary Differential Equations", Addison-Wesley, Reading, 1955.
18. E. SCHWERIN, «Über Spannungen in symmetrisch und unsymmetrisch belasteten Kugelschalen (Kuppeln) insbesondere bei Belastung durch Winddruck», Armierter Beton, v. 12, 1919, pp. 25—37, 54—63, 81—88.
19. E. TORROJA, «Réalisations de voûtes minces en Espagne», Annales de l'Institut Technique du Bâtiment et des Travaux Publics, No. 164, 1950, p. 1.
20. Y. TSUBOI and K. AKINO, "Design and Construction of Reinforced Concrete Spherical Shell of Non-Uniform Thickness Supported on Roller System", Rept. Inst. Indust. Scien., Univ. Tokyo, v. 5, No. 4, 1955, p. 53.
21. I. N. VEKUA, "Integrirovaniye uravneniye sphericheskoy obolochki" (Integral Equations of a Spherical Shell), Prikl. Mat. i Mekh., v. 9, 1945, p. 5.
22. C. S. WHITNEY, "Reinforced Concrete Thin Shell Structures", Jour. A.C.I., v. 24, No. 6, 1953, p. 521.
23. V. Z. VLASOV, "Obshchaya teoriya obolochek i ee prilosheniya v tekhnike" (General Theory Shells and Its Application in Technology), Gostehizdat, Moscow, 1949, p. 295.
24. Y. YOKOWO and O. MATSUOKA, "Approximation of the Bending Theory of a Spherical Shell", Proc. 4th Japan Nat. Congr. Appl. Mech., 1954, p. 177.

Appendix

The sectional quantities (3) can be expressed in terms of solutions (25) and (26):

$$N_{rr} = \frac{pR}{2} + A_0{}^1\left\{\frac{\lambda}{\omega}\frac{1}{r}\mathrm{bei}'(r\lambda)\right\} - A_0{}^2\left\{\frac{\lambda}{\omega}\frac{1}{r}\mathrm{ber}(r\lambda)\right\} +$$

$$+\sum_{n=1}^{\infty}\left\{A_{kn}^1\left[\frac{\lambda}{\omega}\frac{1}{r}\mathrm{bei}_{kn}(r\lambda) - \frac{1}{\omega}\left(\frac{kn}{r}\right)^2\mathrm{bei}_{kn}(r\lambda)\right] + A_{kn}^2\left[-\frac{\lambda}{\omega}\frac{1}{r}\mathrm{ber}'_{kn}(r\lambda) + \right.\right.$$

$$\left.\left. + \frac{1}{\omega}\left(\frac{kn}{r}\right)^2\mathrm{ber}_{kn}(r\lambda)\right] + C_{kn}^2\left[\frac{kn(1-kn)}{\omega}r^{kn-2}\right]\right\}\cos kn\,\theta,$$

$$N_{\theta\theta} = \frac{pR}{2} + A_0{}^1\left\{\frac{\lambda^2}{\omega}\mathrm{bei}''(r\lambda)\right\} - A_0{}^2\left\{\frac{\lambda^2}{\omega}\mathrm{ber}''(r\lambda)\right\} +$$

$$+\sum_{n=1}^{\infty}\left\{A_{kn}^1\left[\frac{\lambda^2}{\omega}\mathrm{bei}''_{kn}(r\lambda)\right] + A_{kn}^2\left[\frac{\lambda^2}{\omega}\mathrm{ber}''_{kn}(r\lambda)\right] + \right.$$

$$\left. + C_{kn}^2\left[\frac{kn(kn-1)}{\omega}r^{kn-2}\right]\right\}\cos kn\,\theta,$$

$$N_{r\theta} = \sum_{n=1}^{\infty} \left\{ A_{kn}^{1} \left[\frac{kn}{r} \frac{\lambda}{\omega} \mathrm{bei}'_{kn}(r\lambda) - \frac{1}{\omega} \frac{kn}{r^2} \mathrm{bei}_{kn}(r\lambda) \right] + A_{kn}^{2} \left[-\frac{\lambda}{\omega} \frac{kn}{r} \mathrm{ber}'_{kn}(r\lambda) + \right.\right.$$

$$\left.\left. + \frac{1}{\omega} \frac{kn}{r^2} \mathrm{ber}_{kn}(r\lambda) \right] + C_{kn}^{2} \left[\frac{kn(kn-1)}{\omega} r^{kn-2} \right] \right\} \sin kn\theta,$$

$$Q_r = D \left[A_0{}^1 \left[-\lambda^3 \mathrm{ber}'''(r\lambda) - \frac{\lambda^2}{r} \mathrm{ber}''(r\lambda) + \frac{\lambda}{r^2} \mathrm{ber}'(r\lambda) \right] + \right.$$

$$+ A_0{}^2 \left[-\lambda^3 \mathrm{bei}'''(r\lambda) - \frac{\lambda^2}{r} \mathrm{bei}''(r\lambda) + \frac{\lambda}{r^2} \mathrm{bei}'(r\lambda) \right] +$$

$$+ \sum_{n=1}^{\infty} \left\{ A_{kn}^{1} \left[-\lambda^3 \mathrm{ber}'''_{kn}(r\lambda) - \frac{\lambda^2}{r} \mathrm{ber}''_{kn}(r\lambda) + \left(\frac{\lambda}{r^2} + \lambda \frac{(kn)^2}{r^2} \right) \mathrm{ber}'_{kn}(r\lambda) - \right.\right.$$

$$- \frac{(kn)^2}{r^3} \mathrm{ber}_{kn}(r\lambda) \right] + A_{kn}^{2} \left[-\lambda^3 \mathrm{bei}'''_{kn}(r\lambda) - \frac{\lambda^2}{r} \mathrm{bei}''_{kn}(r\lambda) + \right.$$

$$\left.\left.\left. + \left(\frac{\lambda}{r^2} + \lambda \frac{(kn)^2}{r^2} \right) \mathrm{bei}'_{kn}(r\lambda) - \frac{(kn)^2}{r^3} \mathrm{bei}_{kn}(r\lambda) \right] \right\} \cos kn\theta \right],$$

$$Q_\theta = D \left[\sum_{n=1}^{\infty} \left\{ A_{kn}^{1} \left[\lambda^2 \frac{kn}{r} \mathrm{ber}'''_{kn}(r\lambda) + \lambda \frac{kn}{r^2} \mathrm{ber}'_{kn}(r\lambda) - \left(\frac{kn}{r} \right)^3 \mathrm{ber}_{kn}(r\lambda) \right] + \right.\right.$$

$$\left.\left. + A_{kn}^{2} \left[\lambda^2 \frac{kn}{r} \mathrm{bei}'''_{kn}(r\lambda) + \lambda \frac{kn}{r^2} \mathrm{bei}'_{kn}(r\lambda) - \left(\frac{kn}{r} \right)^3 \mathrm{bei}_{kn}(r\lambda) \right] \right\} \sin kn\theta \right],$$

$$M_{rr} = D \left[-A_0{}^1 \left\{ \lambda^2 \mathrm{ber}''(r\lambda) + \nu \frac{\lambda}{r} \mathrm{ber}'(r\lambda) \right\} - A_0{}^2 \left\{ \lambda^2 \mathrm{bei}''(r\lambda) + \nu \frac{\lambda}{r} \mathrm{bei}'(r\lambda) \right\} + \right.$$

$$+ \sum_{n=1}^{\infty} \left\{ A_{kn}^{1} \left[-\lambda^2 \mathrm{ber}''_{kn}(r\lambda) + \nu \left(\frac{kn}{r} \right)^2 \mathrm{ber}_{kn}(r\lambda) - \nu \frac{\lambda}{r} \mathrm{ber}'_{kn}(r\lambda) \right] + \right.$$

$$+ A_{kn}^{2} \left[-\lambda^2 \mathrm{bei}''_{kn}(r\lambda) + \nu \left(\frac{kn}{r} \right)^2 \mathrm{bei}_{kn}(r\lambda) - \nu \frac{\lambda}{r} \mathrm{bei}'_{kn}(r\lambda) \right] +$$

$$\left.\left. + C_{kn}^{1} \left[-(1-\nu) kn(kn-1) r^{kn-2} \right] \right\} \cos kn\theta \right],$$

$$M_{\theta\theta} = D \left[A_0{}^1 \left\{ -\frac{\lambda}{r} \mathrm{ber}'(r\lambda) - \nu \lambda^2 \mathrm{ber}''(r\lambda) \right\} + A_0{}^2 \left\{ -\frac{\lambda}{r} \mathrm{bei}'(r\lambda) - \nu \lambda^2 \mathrm{bei}''(r\lambda) \right\} + \right.$$

$$+ \sum_{n=1}^{\infty} \left\{ A_{kn}^{1} \left[\left(\frac{kn}{r} \right)^2 \mathrm{ber}_{kn}(r\lambda) - \frac{\lambda}{r} \mathrm{ber}'_{kn}(r\lambda) - \nu \lambda^2 \mathrm{ber}''_{kn}(r\lambda) \right] + \right.$$

$$+ A_{kn}^{2} \left[\left(\frac{kn}{r} \right)^2 \mathrm{bei}_{kn}(r\lambda) - \frac{\lambda}{r} \mathrm{bei}'_{kn}(r\lambda) - \nu \lambda^2 \mathrm{bei}''_{kn}(r\lambda) \right] +$$

$$\left.\left. + C_{kn}^{1} \left[(1-\nu) kn(kn-1) r^{kn-2} \right] \right\} \cos kn\theta \right],$$

$$M_{r\theta} = D \sum_{n=1}^{\infty} \left[A_{kn}^{1} \left\{ (1-\nu) \left[\lambda \frac{kn}{r} \mathrm{ber}'_{kn}(r\lambda) - \frac{kn}{r^2} \mathrm{ber}_{kn}(r\lambda) \right] \right\} + \right.$$

$$+ A_{kn}^{2} \left\{ (1-\nu) \left[\lambda \frac{kn}{r} \mathrm{bei}'_{kn}(r\lambda) - \frac{kn}{r^2} \mathrm{bei}_{kn}(r\lambda) \right] \right\} +$$

$$\left. + C_{kn}^{1} \{ (1-\nu) [kn (kn-1) r^{kn-2}] \} \right] \sin kn\theta.$$

Summary

An approximate method of analysis for stress and strain in shallow spherical calotte shells has been developed in this paper. The entire problem was reduced to determination of two parametric functions, which describe the overall behavior of thin shell. These functions satisfy exactly the fundamental differential equations of shallow spherical shell, but exhibit prescribed edge conditions only at specified boundary points.

This method of solution is applicable to thin segmental shallow spherical shells (crown height / base circle $\leqq 1/5$) over regular polygonal type base. The boundary conditions are not rigorously satisfied, but for practical purposes they should prove sufficiently accurate. Nevertheless, the numerical computations involved are extremely extensive.

Résumé

L'auteur expose une méthode d'approximation pour le calcul des contraintes et des allongements dans les calottes sphériques minces de faible hauteur. L'ensemble du problème est ainsi ramené à la détermination de deux fonctions paramétriques, qui expriment le comportement global de ces calottes minces. Ces fonctions satisfont aux équations différentielles fondamentales des voûtes sphériques de faible hauteur, mais ne remplissent les conditions marginales prévues qu'en des points limites particuliers.

Cette méthode de résolution peut être appliquée aux calottes sphériques minces de faible hauteur en forme de segment (hauteur de la couronne/cercle de base $\leqq 1/5$), portant sur une base polygonale régulière. Les conditions aux limites ne sont pas rigoureusement satisfaites; la précision obtenue est toutefois suffisante pour les applications de la pratique. Les calculs numériques nécessaires sont néanmoins très compliqués.

Zusammenfassung

In dieser Abhandlung ist eine Näherungsmethode zur Berechnung der Spannungen und Dehnungen in flachen Kugelkalottenschalen entwickelt worden. Das ganze Problem wurde auf die Bestimmung von zwei Parameter-

funktionen, welche das gesamte Verhalten von dünnen Schalen beschreiben, reduziert. Diese Funktionen befriedigen die fundamentalen Differentialgleichungen flacher Kugelschalen genau, erfüllen aber die vorgeschriebenen Randbedingungen nur in einzelnen Grenzpunkten.

Diese Lösungsmethode ist auf dünne, segmentförmige, flache Kugelschalen (Kronenhöhe/Basiskreis $\leqq 1/5$) über regelmäßiger polygonaler Grundfläche anwendbar. Die Randbedingungen werden nicht streng erfüllt; die erreichte Genauigkeit dürfte aber für praktische Zwecke genügen. Dennoch sind die notwendigen numerischen Berechnungen sehr umfangreich.