STRESSES IN ECCENTRICALLY LOADED STEEL COLUMNS

CONTRAINTES DANS LES COLONNES EN ACIER SOLLICITÉES PAR DES CHARGES EXCENTRIQUES

SPANNUNGEN IN EXZENTRISCH BEANSPRUCHTEN STAHLSÄULEN

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1. Introduction.

A slender column, of homogeneous material, perfectly straight and free from initial stresses, subjected to axially applied compressive forces, would not deflect sidewise until the critical value of the load P as given by Euler's Formulae, was reached. Actually, columns as found in Engineering structures are not such ideal columns. Particularly, they do not come under the

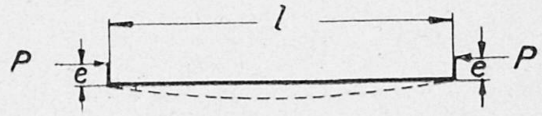

Fig. 1.

heading of "slender" and second the load will not be applied without some eccentricity. As a result of these imperfections the column deflects before the critical value of the load is reached, and the stress for any value of the load is not only a function of the load but also of the deflection of the column.

The true behaviour of such a column will depend upon such indeterminate factors as initial curvature, initial stresses, non-homogenity of material and eccentricity of load. It may be assumed that such a column can be closely approximated by a so called "ideal column", having its load applied with some definite eccentricity (e) the effect of which will be the same as the indeterminate factors mentioned above. Such a column is shown in Fig. 1. An analysis of this case leads to the well known Formula,

$$\sigma_{max} = \frac{P}{A}\left(1 + \frac{e}{k} \sec \frac{l}{2} \sqrt{P/EI}\right) \tag{1}$$

For pin ended columns the value of the eccentricity-ratio $\left(\frac{e}{k}\right)$ is selected to take care of such imperfections as accidental eccentricity, initial curvature, non-homogenity, etc.

There is, however, no reason why it may not include any known eccentricity of load, or bending moments, such as occur at the ends of compression members in all types of rigid frame construction. It is at once recognized that such moments may not necessarily be equal at the two ends of the column. Possible types of such secondary moments which occur in rigid frame construction are shown in Fig. 2 (a) and (b).

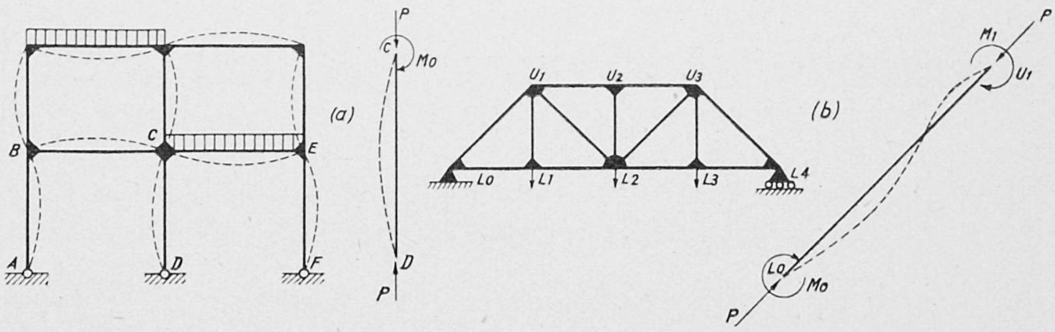

Fig. 2.

As a matter of fact, Equation (1) represents only a special case of eccentric loading, where the end moments are equal and produce the same sign of bending. It is proposed here to develop the more general case, and to present curves for various combinations of loading.

2. General Case.

The general case of a column with unequal eccentricities at the two ends will now be considered. It is assumed that the plane of loading is one of

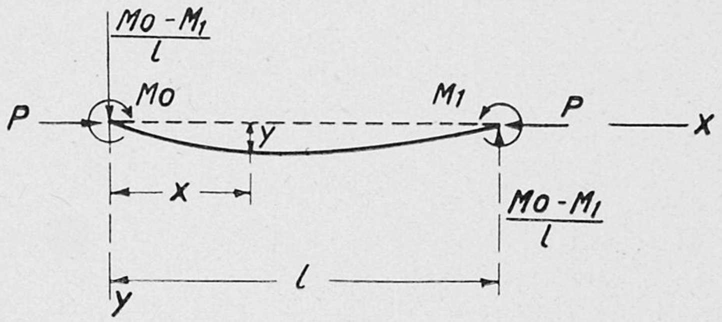

Fig. 3.

the principal planes of the cross-section of the column [1]). Such a column is shown in Fig. 3. M_0 is the numerically larger of two unequal end moments. The bending moment at any section (Fig. 3) will be;

$$M_x = Py + M_0 - \left(\frac{M_0 - M_1}{l}\right) x. \tag{2}$$

The equation of the elastic line then is,

$$EI \frac{d^2y}{dx^2} = -Py - M_0 + \left(\frac{M_0 - M_1}{l}\right) x.$$

The general solution of this equation will be

$$y = C_1 \cos qx + C_2 \sin qx - \frac{M_0}{P} + \frac{M_0 x}{Pl} - \frac{M_1 x}{Pl}$$

where $q = \sqrt{P/EI}$ and C_1 and C_2 are constants of integration.

[1]) The possibility of buckling in the plane perpendicular to the plane of loading is not considered.

To satisfy end conditions we have,

$$C_1 = \frac{M_0}{P}$$

and

$$C_2 = \frac{M_1}{P \sin ql} - \frac{M_0}{P \tan ql}$$

Then:

$$y = \frac{M_0}{P} \cos qx + \frac{M_1 \sin qx}{P \sin ql} - \frac{M_0 \sin qx}{P \tan ql} - \frac{M_0}{P} + \frac{M_0 x}{Pl} - \frac{M_1 x}{Pl} \tag{3}$$

Taking the first derivative of this with respect to x gives,

$$\frac{dy}{dx} = -\frac{q M_0}{P} \sin qx + \frac{q M_1 \cos qx}{P \sin ql} - \frac{q M_0 \cos qx}{P \tan ql} + \frac{M_0}{Pl} - \frac{M_1}{Pl} \tag{4}$$

and from this equation the slope at the left end of the column will be,

$$\left(\frac{dy}{dx}\right)_{x=0} = \frac{q M_1}{P \sin ql} - \frac{q M_0}{P \tan ql} + \frac{M_0}{Pl} - \frac{M_1}{Pl} \tag{5}$$

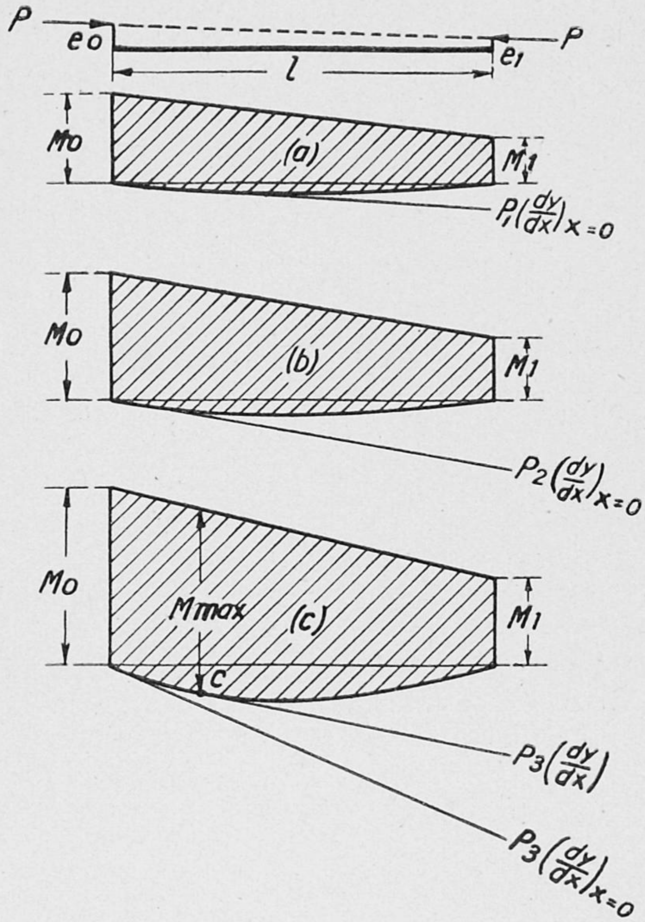

Fig. 4.

In order to find the maximum stress in the column, attention must be directed to the position of maximum bending moment. Referring to Fig. 4 which represents the beding moment diagram for three different values of the load P designated by P_1, P_2, and P_3; P_1 being the smallest, and P_3 the greatest value, it is seen that so long as the slope at the left end of that part of the moment diagram due to Py, is less than the slope $\frac{M_0 - M_1}{l}$ of that part of the diagram due to M_0 and M_1, M_0 will be the maximum ordinate in the diagram. This condition is represented by Fig. 4 (a). On the other hand when the load has some value P_3 for which the slope at the left end, $P_3\left(\frac{dy}{dx}\right)_{x=0}$ is greater than the slope $\frac{M_0 - M_1}{l}$ then the maximum ordinate will occur at some point C, (Fig. 4 c) namely where the slope $P_3\left(\frac{dy}{dx}\right) = \frac{M_0 - M_1}{l}$. Obviously there will be some value of the load P_2 where the slope at the left end, $P_2\left(\frac{dy}{dx}\right)_{x=0}$ will just equal the slope $\frac{M_0 - M_1}{l}$ (Fig. 4 b). This value P_2, then, marks a certain value of the load, below which the maximum bending moment will always occur at the left end and have a value simply of M_0, and above which the maximum bending moment will occur at some point in the column and have a value greater than M_0.

The calculation of maximum stress will consist of two steps. First to find the value of this load P_2 and second, to find the value of x at which this greater maximum bending moment occurs, and evaluate this maximum from equation (2).

To find the value of the load (P_2) the following equation must be satisfied.

$$P\left(\frac{dy}{dx}\right)_{x=0} = \frac{M_0 - M_1}{l}.$$

Substituting the value of $\left(\frac{dy}{dx}\right)_{x=0}$ from equation (5) gives

$$P\left[\frac{q\,M_1}{P\sin ql} - \frac{q\,M_0}{P\tan ql} + \frac{M_0}{Pl} - \frac{M_1}{Pl}\right] = \frac{M_0}{l} - \frac{M_1}{l},$$

which reduces to,

$$\frac{M_1}{M_0} = \cos ql.$$

Remembering that $M_1 = Pe_1$, $M_0 = Pe_0$, $q = \sqrt{P/EI}$, and calling the ratio $e_0/e_1 = \alpha$, the load at which the maximum bending moment first becomes larger than M_0 will be, from the preceeding equation,

$$P_2 = \frac{(\cos^{-1}\alpha)^2 EI}{l^2} \tag{6}$$

and the corresponding average load will be

$$\frac{P_2}{A} = \frac{(\cos^{-1}\alpha)^2 E}{(l/r)^2} \tag{7}$$

If the load is larger than P_2, the maximum bending moment occurs at a point (Fig. 4 c) where

$$P\left(\frac{dy}{dx}\right) = \frac{M_0 - M_1}{l}.$$

Using equation (4) we find,

$$P\left[-\frac{q\,M_0}{P}\sin qx + \frac{q\,M_1\cos qx}{P\sin ql} - \frac{q\,M_0\cos qx}{P\tan ql} + \frac{M_0}{Pl} - \frac{M_1}{Pl}\right] = \frac{M_0}{l} - \frac{M_1}{l}$$

which reduces to,

$$\tan qx = \frac{e_1}{e_0}\csc ql - \cot ql \tag{8}$$

This equation gives the value of x for maximum bending moment for all values of the load greater than P_2. Placing the expression for (y) from equation (3) in the general expression for bending moment, (equation 2) gives

$$M_x = P\left[\left(\frac{e_1}{\sin ql} - \frac{e_0}{\tan ql}\right)\sin qx + e_0\cos qx\right] \tag{9}$$

To make this a maximum the expression for x from equation 8 will be used giving,

$$M_{max} = Pe_0\sqrt{\alpha^2 - 2\alpha\cos(l\cdot\sqrt{P/EI}) + 1}\cdot\csc\left(l\sqrt{P/EI}\right) \tag{10}$$

This holds true for all values of the load greater than P_2.

It will be seen that if α is made equal to unity, which would be the case shown in Fig. 1, equation (10) will reduce to the well known equation,

$$M_{max} = Pe\sec\left(\frac{l}{2}\sqrt{P/EI}\right).$$

Using for the numerical maximum of stress the known equation,

$$\sigma_{max} = \frac{P}{A} + \frac{M_{max}}{S}$$

we find for values of the load less than P_2, in which case $M_{max} = M_0$,

$$\sigma_{max} = \frac{P}{A}\left(1 + \frac{e_0}{k}\right) \tag{11}$$

while for values greater than P_2, in which case M_{max} is given by Equation (10)

$$\sigma_{max} = \frac{P}{A}\left[1 + \frac{e_0}{k}\sqrt{\alpha^2 - 2\alpha\cos\left(\frac{l}{r}\sqrt{P/AE}\right) + 1}\cdot\csc\left(\frac{l}{r}\sqrt{P/AE}\right)\right] \tag{12}$$

These last two equations give the relation between maximum stress, average unit load, and the dimensions of the column, for all possible values of the load, while equation 7 gives the particular value of the load P_2, which marks the point of transition between the two equations. In other words, equation (7) gives the only value of the load for which both equations (11) and (12) give the same stress.

In order to make these three equations applicable for design purposes a factor of safety will be applied to them. This factor will be based on the yield point stress of the material, on the basis that when any part of the column has reached this yield point stress the column will suffer some permanent deformation and therefore be damaged.

It will be noted that equation (12) does not represent a straight line relationship between load and maximum stress. Hence the factor of safety will be incorporated directly into these equations, by setting $\sigma_{max} = \sigma_{yp}$ and ad the same time replacing P by nP. Equations (11), (12) and (7) then become:

$$\frac{P}{A} = \frac{\sigma_{yp}}{n + ne_0/k} \tag{13}$$

$$\frac{P}{A} = \frac{\sigma_{yp}}{n + (ne_0/k)\sqrt{\alpha^2 - 2\alpha\cos\left[(l/r)\sqrt{np/AE}\right] + 1}\left[\csc\left((l/r)\sqrt{np/AE}\right)\right]} \tag{14}$$

$$\frac{P_2}{A} = \frac{(\cos^{-1}\alpha)^2 E}{n\,(l/r)^2} \tag{15}$$

Equation (13) or (14) will now give a value of the load such that n times this load will always exactly produce a maximum fibre stress equal to the yield point stress of the material.

From these equations the safe load can be determined, for any given column and given value of $\alpha = \dfrac{e_0}{e_1}$. The solution of these equations however must be made of a cut and try method. In the case of equation (13) such a

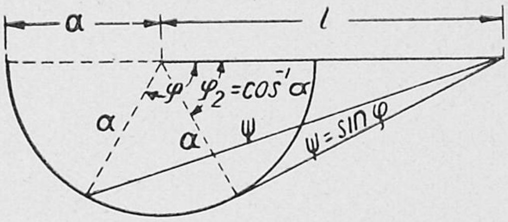

Fig. 5.

solution presents no difficulty, but in the case of equation (14) some further explanation of a method of solution is necessary. The problem is further complicated by the fact that at the beginning it is not known whether equation (13) or equation (14) is going to apply.

In order to simplify equation (14) let the quantity,

$$\left[(l/r)\sqrt{nP/AE}\right] = \varphi,$$

and let the quantity,

$$\sqrt{\alpha^2 - 2\alpha\cos\varphi + 1} = \psi.$$

Equation (14) then becomes

$$\frac{P}{A} = \frac{\sigma_{yp}}{n + (ne_0/k)(\psi\csc\varphi)}. \tag{16}$$

From equations (13) and (16) guided by equation (15) curves may be plotted showing the relation between allowable average compressive load $\dfrac{P}{A}$ and the slenderness-ratio $\dfrac{l}{r}$ of the column for any ratio α from -1 to $+1$, which are the limits of its variation.

Since, however, in equation (16) the quantity ψ has both α and φ involved in it, some preliminary evaluation of this quantity for any given value of α is necessary in order to make the solution of equation (16) possible. If attention is directed to the quantity $\psi = \sqrt{\alpha^2 - 2\alpha \cos \varphi + 1}$, it is seen to represent the third side of a triangle when two sides and the included angle are known. This is shown in Fig. 5. In this case the two sides are pure numbers, α and 1, and the included angle is φ. Referring to Fig. 5 it is seen that for any given value of α, ψ varies from $1 - \alpha$ when $\varphi = 0$, to $1 + \alpha$ when $\varphi = \pi$. For any given value of α, then ψ may be evaluated for several values of φ and then a curve plotted showing values of the quantity ($\psi \csc \varphi$) in equation (16), against values of φ, (note that φ like (q) is a function of the load P). Such curves have been plotted for values of α ranging by .25 intervals from -1 to $+1$. (See Fig. 6.)

Equation (15) may be expressed as $\frac{r}{l}\sqrt{nP_2/AE} = \cos^{-1}\alpha = \varphi_2$ from which it is seen that equation (16) does not hold for values of φ less than $\cos^{-1}\alpha$.

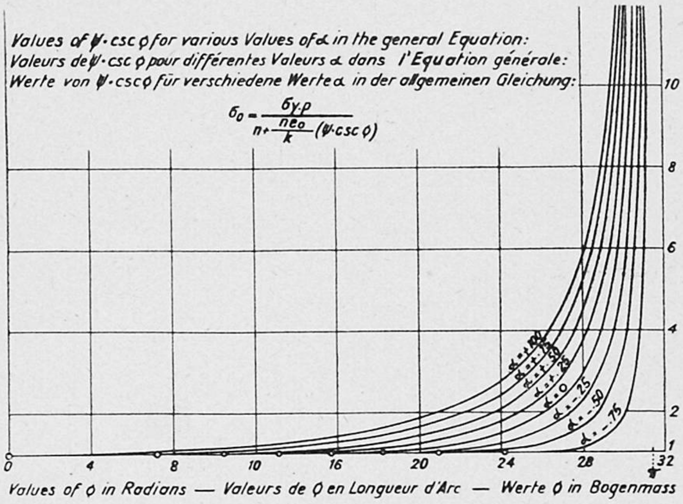

Fig. 6.

By the use of the curves in Fig. 6, equation (16) may be readily solved. Now with equations (13) and (16), separated by equation (15), curves showing average allowable load $\left(\frac{P}{A}\right)$ against slenderness ratio $\left(\frac{l}{r}\right)$ have been plotted for each of the values of α given in Fig. 6, with the idea that results may be interpolated from these sets for any value of α. These curves are shown in Figures 7 to 15.

A study of these curves (particularly the position of the curve given by equation (15)) will show clearly how as α decreases from $+1$ to -1 the range of $\frac{l}{r}$ in which the allowable average compressive load is a straight horizontal line, independent of the deflection of the column, increases until when $\alpha = -1$, the column behaves exactly like an axially loaded one, (i. e. Equation (15) coincides with the Euler curve).

The moments at the ends of members in rigid frame construction, arising from the rigidity of the joints will generally be such as to make the $\frac{e_0}{k}$ ratio

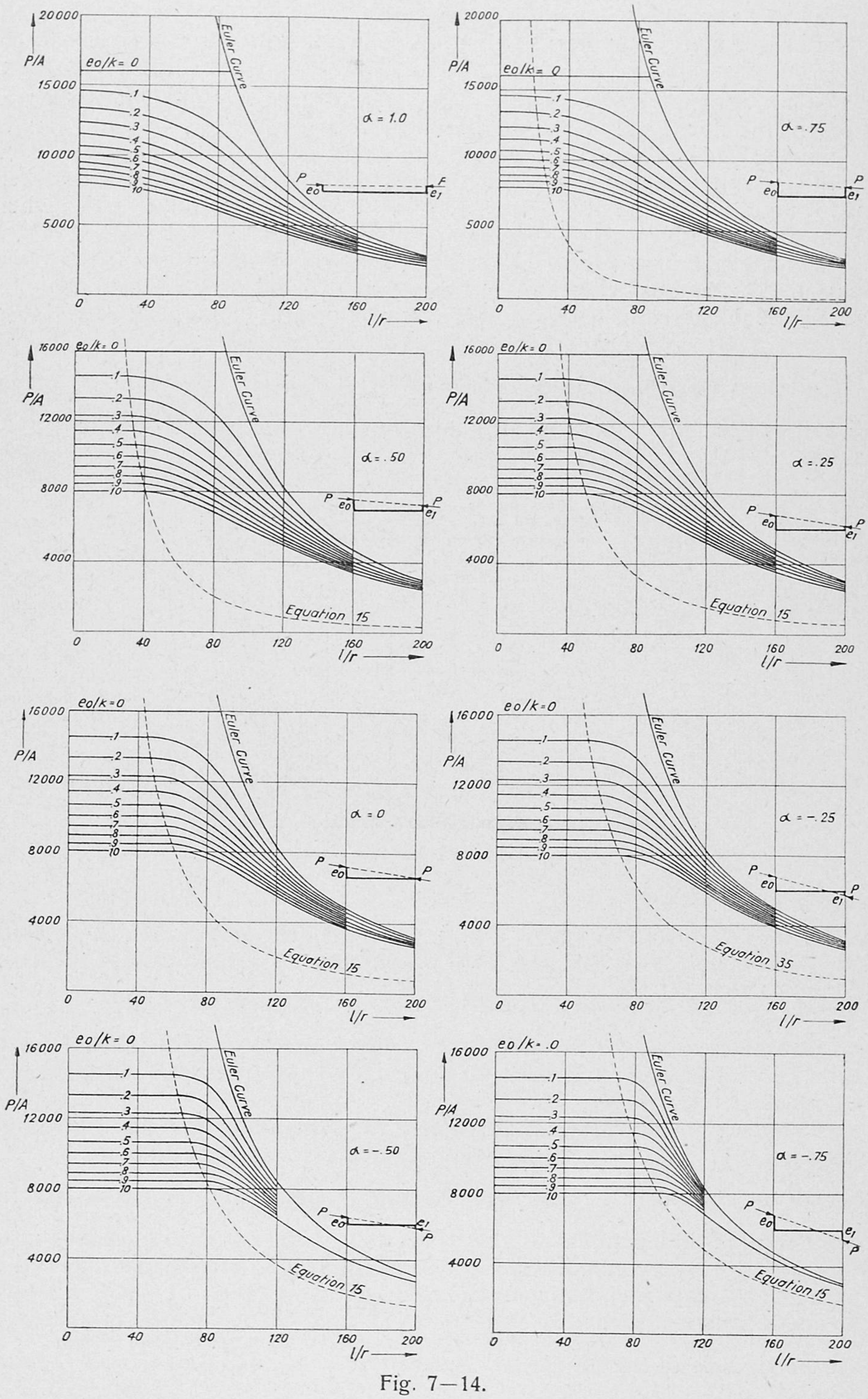
P/A
l/r
Euler Curve
eo/k = 0
.1
.2
.3
.4
.5
.6
.7
.8
.9
10
α = 1.0
α = .75
α = .50
α = .25
α = 0
α = −.25
α = −.50
α = −.75
Equation 15
Equation 35
P
e0
e1
20000
15000
10000
5000
16000
12000
8000
4000
0
40
80
120
160
200

Fig. 7—14.

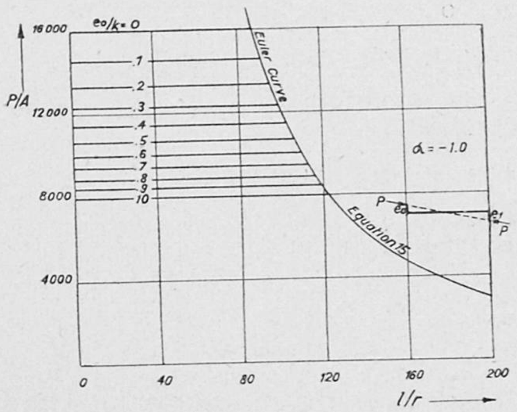

Fig. 15.

Fig. 7—15:

Graphiques des efforts de bordure pour des rapports $\frac{e_0}{k} = 0.1$ à 1.0 (Eq. 16); degré de sécurité par rapport à la limite d'écoulement du materiau ($\sigma_{yp} = 2800$ kg/cm²) n = 2.5; $\alpha = \frac{e_1}{e_0}$ variant entre $+1.0$ et -1.0. Ordonnées en livres par pouce carré.

Graphische Tabellen der Randspannungen für Werte $\frac{e_0}{k}$ von 0.1 bis 1.0 (siehe Gl. 16); Sicherheitsgrad gegen Erreichen der Streckgrenze ($\sigma_{yp} = 2800$ kg/cm²) n = 2.5; $\alpha = \frac{e_1}{e_0}$ für $+1.0$ bis -1.0. Ordinaten in Pfund je Quardratzoll.

Design curves of fibre stress for ratios of $\frac{e_0}{k} = 0.1$ to 1.0 (Eq. 16); degree of security against reaching of yield point stress ($\sigma_{yp} = 40.000$ lbs./sq. in.) n = 2.5; $\alpha = \frac{e_1}{e_0}$ varying from $+1.0$ to -1.0. Ordinates in lbs/sq. in.

lie between 0 and 1 (i. e. the secondary stress will usually be less than 100% of the primary stress). For this reason the curves (Figs. 7 to 15) have been plotted for $\frac{e_0}{k}$ ratios from 0 to 1, going by $^1/_{10}$th intervals.

Notations used.

P = a compressive load on the column
A = the cross-sectional area of the column
E = modulus of elasticity of the material
I = moment of inertia of the cross-section of the column
l = length of the column
$\frac{P}{A}$ = average compressive load on the column
S = section modulus of the cross-section of the column
r = radius of gyration of the cross-section
q = $\sqrt{P/EI}$ = a funktion of the load P
$\frac{l}{r}$ = slenderness-ratio of the column
M = bending moment
e = eccentricity of the load P
k = $\frac{S}{A}$ = core distance of the cross section
σ = fibre stress
n = a factor of safety

Signes utilisés — Angewandte Bezeichnungen.

P = charge de la colonne	Druckkraft der Säule
A = section de la colonne	Fläche der Säule
E = module d'élasticité de la colonne	Elastizitätsmodul der Säule
I = moment d'inertie de la colonne	Trägheitsmoment der Säule
l = longueur de la colonne	Säulenlänge
$\frac{P}{A}$ = pression spécifique de la colonne	spez. Pressung der Säule
S = moment de résistance de la colonne	Widerstandsmoment der Säule
r = rayon d'inertie de la colonne	Trägheitsradius der Säule

$q = \sqrt{P/EI}$ = fonction de la force P	Funktion der Kraft P
$\frac{l}{r}$ = degré d'élancement	Schlankheitsgrad der Säule
M = moment de flexion	Biegungsmoment
e = excentricité de P	Exzentrizität von P
$k = \frac{S}{A}$ = diamètre du noyau central	Kernweite des Querschnitts
σ = contrainte	Spannung
n = coefficient de sécurité	Sicherheitsfaktor

Summary.

By use of the curves (Figures 7—15) compression members in rigid frame construction may be designed with a definite factor of safety, based on the yield-point load, once the secondary moments originating from the rigidity of the joints are known.

The cut and try method of solution can be performed very rapidly by use of the curves. As will be seen, these curves have been drawn for a structural steel (σ_{yp} = 40 000 lbs. per sq. in.) and a factor of safety $n = 2.5$ which puts them on about the same basis with standard American specifications for column formulae.

No attempt has been made here to consider such important problems, as buckling beyond the yield point, eccentricity of load not in a principal plane, and transverse loads.

Résumé.

A l'aide des courbes des figures 7—15 il est possible de déterminer les dimensions des éléments comprimés excentriquement tel que, par exemple, les montants des cadres de stabilité, dans lesquels leur assemblage rigide avec les poutres transversales et les entretoises de contreventement donne naissance à des moments fléchissants différents aux deux extrémités des montants. Le coefficient de sécurité est déterminé par rapport à la limite d'écoulement du materiau.

La détermination des tensions à l'aide des graphiques se fait très rapidement. Les courbes ont été calculées pour de l'acier de construction (limite d'écoulement σ_{yp} = 2800 kg/cm^2), en tenant compte d'un coefficient de sécurité $n = 2.5$, ce qui correspond à peu près aux prescriptions gouvernementales des Etats Unis.

Plusieurs problèmes importants n'ont pas encore été pris en considération, soit: Flambage au-dessus de la limite d'écoulement, forces agissant en dehors du plan principal (c'est-à-dire agissant dans les trois dimensions), forces de cisaillement.

Zusammenfassung.

Mittels der Kurven (Fig. 7—15) können Druckglieder in steifen Rahmenkonstruktionen mit einem festen Sicherheitsfaktor, der sich auf die Fließgrenze stützt, berechnet werden, sobald die Biegungsmomente, die von der Steifigkeit der Knoten herrühren, bekannt sind.

Die Lösung mittels Versuchsrechnung erfolgt sehr rasch unter Zuhilfenahme der Kurven. Diese Kurven wurden gezeichnet für Baustahl (σ_{yp} = 2800 kg/cm^2) mit einem Sicherheitsfaktor n = 2,5, was ungefähr den amerikanischen Vorschriften entspricht.

Verschiedene wichtige Probleme sind bis anhin noch nicht in Betracht gezogen worden: Knicken jenseits der Fließgrenze, excentrischer Lastangriff außerhalb der Hauptebene, Querkräfte.

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