Determination of the shape of fixed-ended beams for maximum economy according to the plastic theory

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Determination of the shape of fixed-ended beams for maximum economy according to the plastic theory

Détermination de la forme à donner aux poutres encastrées d'après la théorie de la plasticité en vue du maximum d'économie

Bestimmung der wirtschaftlichsten Querschnittsform eingespannter Balken nach der Plastizitätstheorie

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1. Introduction

In the design of structures according to the plastic theory, the members are so proportioned that collapse would not occur at a load less than the working load multiplied by a "load factor." The plastic theory provides a means of estimating the collapse loads of ductile structures by considering their behaviour beyond the elastic limit. It has been shown[1] that, in the absence of instability, these collapse loads may be calculated simply by reference to the conditions of equilibrium, without considering the equations of flexure. Hence the design process is essentially reduced to the selection of members with plastic moments of resistance sufficient to withstand the bending moments imposed by the "factored loads"—that is, by the working loads multiplied by the load factor.

The direct nature of the design of structures by the plastic theory facilitates the relative proportioning of the members such that the total weight is an absolute minimum. A method of proportioning simple structures composed of prismatic members for minimum weight has already been presented.[2] Further economy of material can, however, be achieved by using members of varying cross-section, and may be sufficient to compensate for the increased cost of fabrication. It is thus worth while investigating the maximum saving in material theoretically attainable by this means. No consideration will be given to the increased cost of manufacture of such members compared with those of uniform section, since this must depend primarily on the quantities required; for this reason, it is impossible to arrive at any conclusions regarding possible overall economies.

[1] For references see end of paper.

The relationship to be assumed between weight per unit length and full plastic moment of resistance is discussed in 2 below; 3 contains a discussion of a member of continuously varying section fixed at the ends and supporting a uniformly distributed load; while the case of a similarly loaded member in which the cross-section is only to be varied by one or two discrete intervals is discussed in 4.

The term "fixed at the ends" is not here intended to imply complete flexural rigidity at the supports, but rather that the members to which the beam under consideration is attached are together capable of resisting the full plastic moment of the end sections of that beam.

2. The relationship between full plastic moment and weight per unit length

The full plastic moment of a member (denoted by M_p) is the moment of resistance when the whole section is undergoing plastic deformation. If f_y is the yield stress, at which pure plastic deformation can occur, then for a beam of rectangular cross-section, of width b and depth $2d$ (see fig. 1),

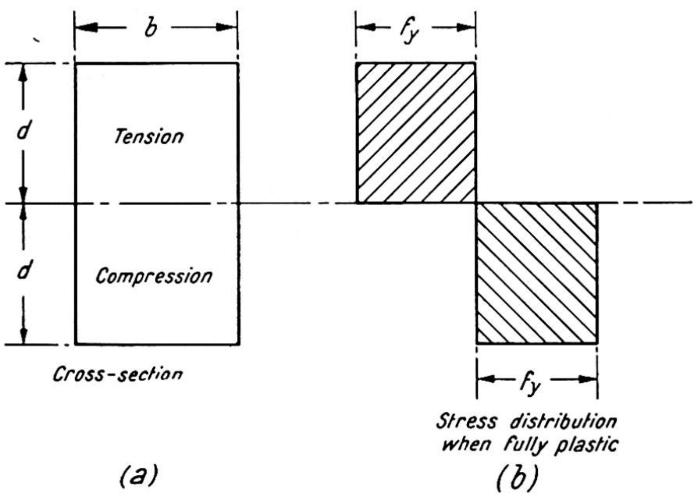

Fig. 1. Fully plastic stress distribution for a rectangular beam

$$M_p = bd^2 f_y \qquad (1)$$

Let the weight per unit length of the beam be w, and let the density of the material be ρ. Then

$$w = 2bd\rho \qquad (2)$$

If b is constant and d varies, then

$$w \propto M_p^{\frac{1}{2}} \qquad (3)$$

If d is constant and b varies,

$$w \propto M_p \qquad (4)$$

while if b and d both vary such that b/d remains constant

$$w \propto M_p^{\frac{2}{3}} \qquad (5)$$

Hence, however the section is varied,

$$w = kM_p^n \qquad (6)$$

where k is a constant and $\frac{1}{2} < n < 1$.

Arguments similar to the above may be applied to sections other than rectangular, and thus equation (6) gives a general relationship between M_p and w. This formula isfactory in that it takes no account of the effect of shear forces. hear forces have little effect on the value of the full plastic moment,[3] esisting shear forces will prevent the section of a beam being allowed the applied bending moment at collapse is zero. Hence it will in ned that

$$w=w_0+kM_p^n \quad \text{. . . . .} \quad (7)$$

constant.

ENDED BEAM OF CONTINUOUSLY VARYING SECTION

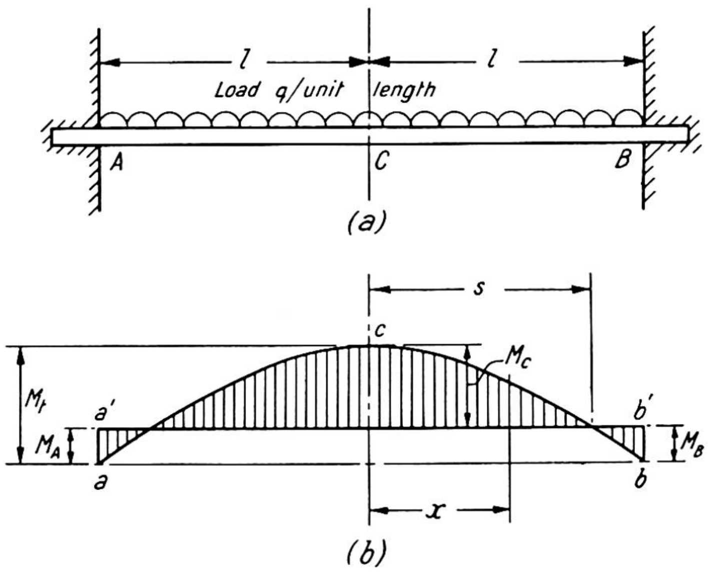

Fig. 2. Bending moment distribution for a beam of continuously varying section (uniformly distributed load)

The beam AB (see fig. 2(*a*)), of length $2l$, is fixed at the ends and carries a uniformly distributed load at collapse of q per unit length. Let the hogging bending moments at the ends (M_A and M_B) be assumed equal at collapse, and let M_C denote the sagging bending moment at the centre. Let M_t be the central bending moment which would be induced in a similar simply supported beam. The bending moment distribution at collapse in the fixed-ended beam may be obtained by superimposing on a parabolic bending moment diagram acb of height M_t (fig. 2(*b*)) the bending moment distribution aa′b′a due to the terminal moments M_A and M , giving the resultant shaded area. Let s denote the distance of the points of contraflexure from the centre of length of the beam.

Then

$$\left.\begin{aligned} M_t &= \frac{ql^2}{2} \\ M_A = M_B &= \frac{l^2-s^2}{l^2} M_t \\ M_C &= \frac{s^2}{l^2} M_t \end{aligned}\right\} \quad \text{. . . . .} \quad (8)$$

If x denotes the distance of any section from the centre of length of
then the minimum full plastic moment at that section becomes

$$\left.\begin{aligned} &\text{when } 0<x<s, & M_p&=\frac{s^2-x^2}{l^2}M_t \\ &\text{when } s<x<l, & M_p&=\frac{x^2-s^2}{l^2}M_t \end{aligned}\right\} \quad . \; . \; . \; . \; . \; .$$

Hence if W denotes the weight of the beam,

$$W=2w_0l+2k\frac{M_t^n}{l^{2n}}\left[\int_o^s(s^2-x^2)^n dx+\int_s^l(x^2-s^2)^n dx\right] \quad . \; .$$

The most economical design will be obtained with that value of s for which W
minimum, i.e. putting $dW/ds=0$, when

$$\int_o^s(s^2-x^2)^{n-1}dx=\int_s^l(x^2-s^2)^{n-1}dx \quad . \; . \; . \; . \; . \; (11)$$

If M_p' and W' denote the full plastic moment and weight respectively of the least
prismatic beam sufficient to carry the load, then

$$M_p'=\frac{M_t}{2} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; (12)$$

$$W'=2w_0l+2k\left(\frac{M_t}{2}\right)^n l \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; (13)$$

When $n=0{\cdot}5$, the most economical value of s is given by

$$\frac{s}{l}=\operatorname{sech}\frac{\pi}{2}=0{\cdot}3986$$

The corresponding minimum weight is

$$W=2w_0l+0{\cdot}9172kM_t^{\frac{1}{2}}l$$

while $$W'=2w_0l+1{\cdot}4142kM_t^{\frac{1}{2}}l$$

The percentage saving of material depends on the ratio of w_0 to $kM_t^{\frac{1}{2}}$. If the requirements of resistance to shear are ignored ($w_0=0$), an economy of up to 35·1% of the weight of the uniform beam can be achieved. When the effect of shear is allowed for, the percentage economy will become less.

When $n=1{\cdot}0$, the economical value of s is $s/l=0{\cdot}5$,

whence $$W=2w_0l+0{\cdot}5kM_tl$$

while $$W'=2w_0l+kM_tl$$

In this case therefore a maximum economy (ignoring shear) of 50% is possible.

When $\frac{1}{2}<n<1$, it may be shown from equation (11) that the most economical value of s is given approximately by the formula

$$\frac{l}{s}=2+1{\cdot}2467\left(\frac{3}{2}\right)^{1-n}\left(\frac{1-n}{1+n}\right) \quad . \; . \; . \; . \; . \; . \; . \; (14)$$

Values of s/l for various values of n are given in Table I.

TABLE I

n	s/l
1·0	0·5000
0·9	0·4835
0·8	0·4651
0·7	0·4447
0·6	0·4226
0·5	0·3986

Points of contraflexure for beam of continuously varying section carrying a uniform load (see fig. 2)

Although for any given value of n the maximum economy is only achieved for some definite value of s, the loss in economy is negligible if $s/l=0{\cdot}45$. This is demonstrated in fig. 3, which shows the percentage economies achieved (assuming $w_0=0$) with various values of s/l for $n=0{\cdot}5$ and $n=1{\cdot}0$.

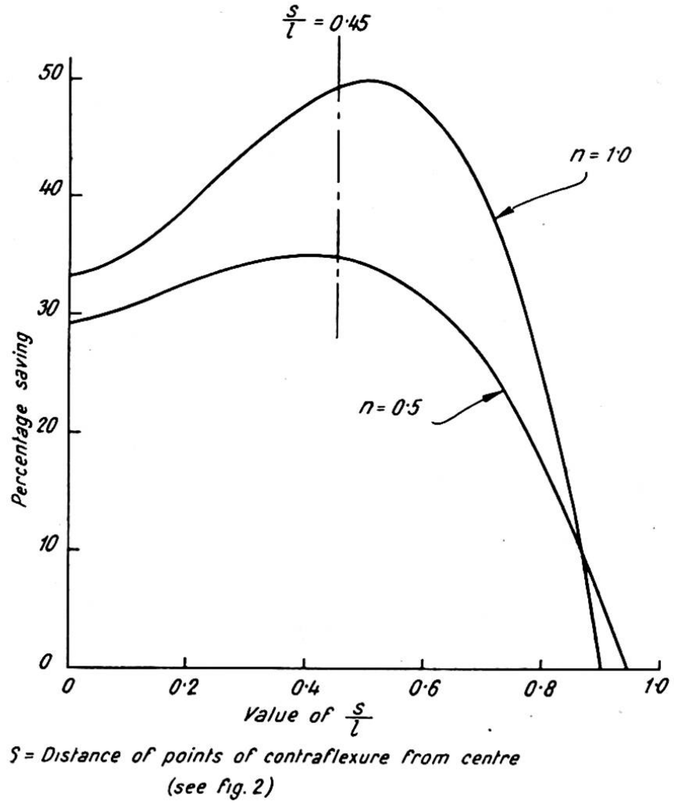

Fig. 3. Economies achieved by continuously varying the section of a fixed-ended beam (uniformly distributed load)

4. FIXED-ENDED BEAM WITH DISCRETE VARIATIONS IN SECTION

Due to the practical difficulties of varying the section of a beam continuously as envisaged above, it is worth while investigating the economies which can be achieved when the full plastic moment is increased by discrete amounts (a) at the centre only, (b) at the ends only and (c) at both centre and ends.

Since the full plastic moment of resistance is nowhere reduced to zero, there will in general be no need to allow for the effects of shear on the relationship between w and M_p (equation (7)). In the following analysis it is therefore assumed that $w_0=0$.

(a) *Section increased over a central length only*

Let the beam previously considered have a uniform value of M_p denoted by M_1, except over a central length $2a$, where it is reinforced so that $M_p = M_2$ where $M_2 > M_1$ (see fig. 4(a)). The bending moments at collapse are shown by the shaded area in

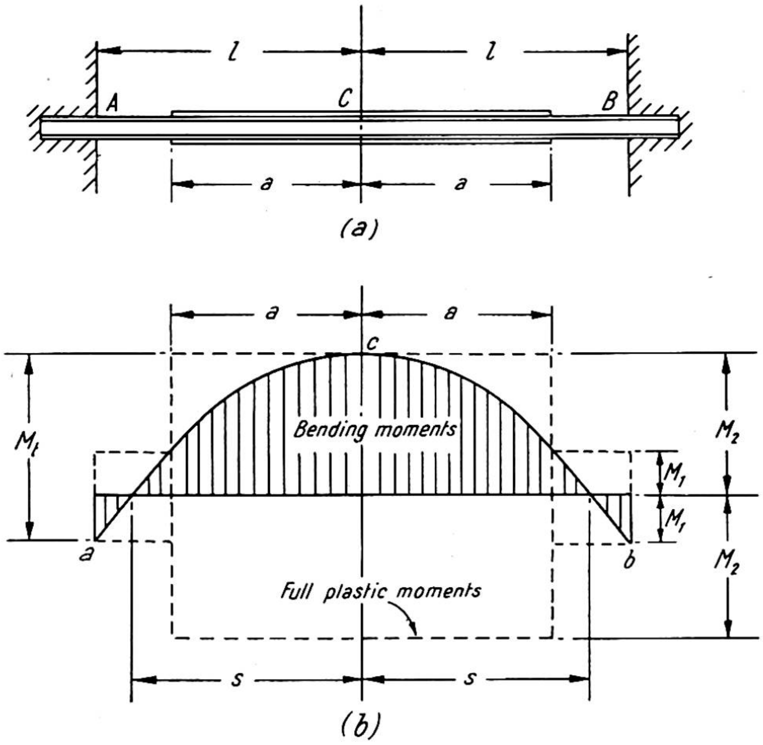

Fig. 4. Bending moment distribution for a beam reinforced at centre only (uniformly distributed load)

fig. 4(b), while the moments of resistance M_1 and M_2 are indicated by dotted lines, which must completely enclose the bending moment diagram. Hence

$$M_1 = M_A = M_B = \frac{l^2 - s^2}{l^2} M_t \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

$$M_2 = M_C = \frac{s^2}{l^2} M_t \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

The value of a is obtained by noting that where the beam changes section, the sagging moment is equal to M_1, and hence

$$M_1 = \frac{s^2 - a^2}{l^2} M_t \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

It follows from equations (15) and (16) that

$$M_1 + M_2 = M_t \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

while from equations (15) and (17), putting $M_1/M_t = r$,

$$\frac{a}{l} = \sqrt{1 - 2r} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

The total weight W of the beam is given by

$$W = 2kM_1{}^n(l - a) + 2kM_2{}^n a \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

It may be shown from equations (18), (19) and (20) that

$$W=2kM_t^{\,n}l[r^n(1-\sqrt{1-2r})+(1-r)^n\sqrt{1-2r}] \quad \dots \quad (21)$$

When W has its minimum value,

$$\left(\frac{r}{1-r}\right)^{1-n}=\frac{n\sqrt{1-2r}+(2nr-n+r)}{1-(2nr-n+r)} \quad \dots\dots \quad (22)$$

TABLE II

n	$r=M_1/M_t$	a/l
1·0	0·4444	0·3333
0·9	0·4432	0·3371
0·8	0·4418	0·3412
0·7	0·4403	0·3456
0·6	0·4388	0·3500
0·5	0·4370	0·3550

Plastic moment ratio and proportion of beam to be reinforced for beam reinforced at centre only (see fig. 4)

The most economical values of r and a/l are given in Table II for values of n between 0·5 and 1·0. It may be noted that r represents the ratio of M_1, the full plastic moment of the unreinforced part of the beam, to M_t, the full plastic moment of the uniform simply supported beam which would just carry the same load. Hence r will be termed the "plastic moment ratio." It will be seen that r and a/l (the proportion of the beam to be reinforced) are almost constant, r varying from 0·4444 to 0·4370 and a/l from 0·3333 to 0·3550. As a working rule therefore the beam should be reinforced for about one-third of its length, the reinforced section having a full plastic moment some 25% or 30% greater than the unreinforced section.

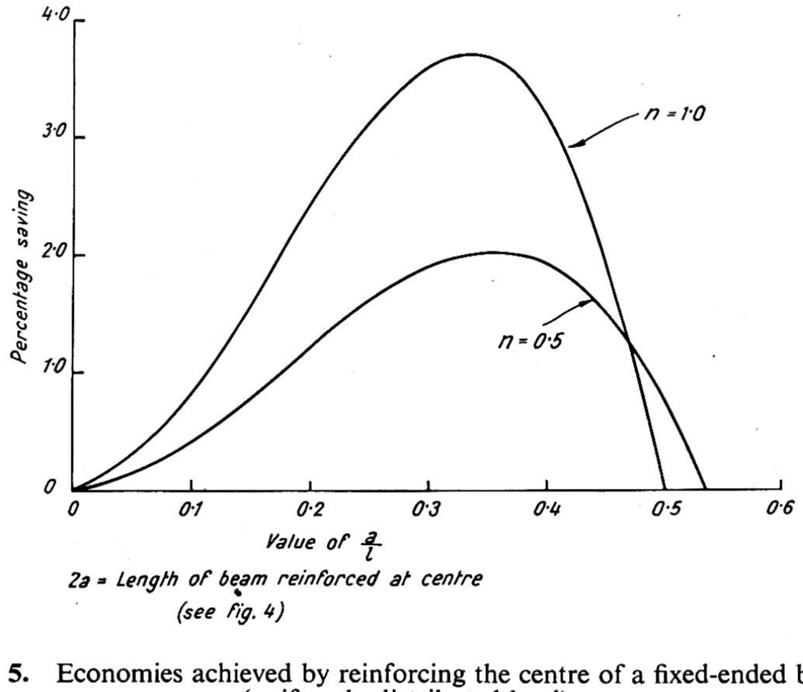

Fig. 5. Economies achieved by reinforcing the centre of a fixed-ended beam (uniformly distributed load)

The variation of the percentage saving (as compared with a beam of uniform section throughout) with a/l for $n=0{\cdot}5$ and $n=1{\cdot}0$ is shown in fig. 5. It will be observed that if more than about half of the beam is reinforced, there is no saving in material. When $n=0{\cdot}5$, the maximum saving possible is $2{\cdot}03\%$ as compared with a saving of $35{\cdot}1\%$ when the section is varied continuously in an ideal manner. When $n=1{\cdot}0$, the corresponding figures are $3{\cdot}70\%$ and $50{\cdot}0\%$ respectively. It is therefore apparent that no great advantage accrues by increasing the section only at the centre.

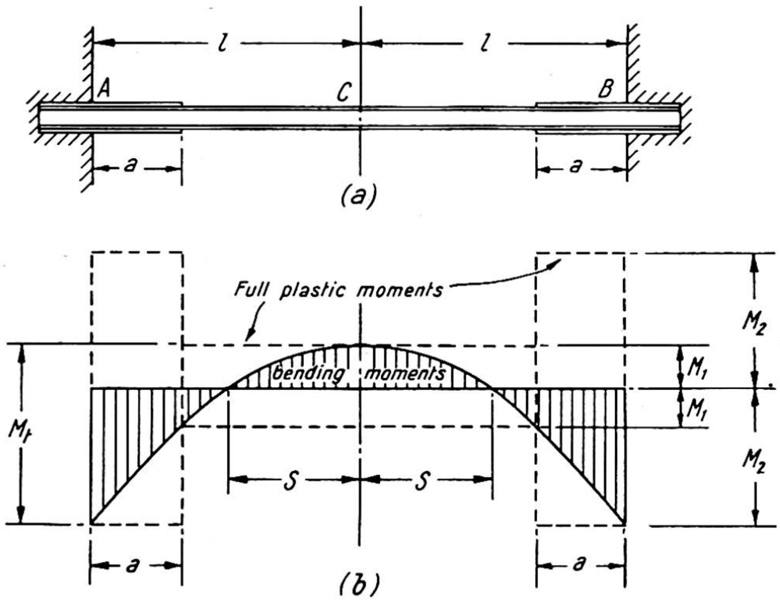

Fig. 6. Bending moment distribution for a beam reinforced at ends only (uniformly distributed load)

(b) *Section increased at ends only*

A beam of uniform plastic moment of resistance M_1 is reinforced for a distance a from either end so that its plastic moment of resistance becomes M_2 (see fig. 6(*a*)). The bending moment distribution at collapse is shown by the shaded area in fig. 6(*b*), while the moments of resistance are superimposed as dotted lines. If full plastic moments are just sufficient to withstand the applied moments, then

$$M_1=M_C=\frac{s^2}{l^2}M_t \quad . \quad . \quad . \quad (23)$$

$$M_2=M_A=M_B=\frac{l^2-s^2}{l^2}M_t \quad . \quad . \quad . \quad (24)$$

Since where the beam changes section the hogging moment has the value M_1,

$$M_1=\frac{(l-a)^2-s^2}{l^2}M_t \quad . \quad . \quad . \quad (25)$$

From equations (23) and (24),

$$M_1+M_2=M_t \quad . \quad . \quad . \quad (26)$$

while from equations (23) and (25), if $M_1/M_t=r$,

$$\frac{a}{l}=1-\sqrt{2r} \quad . \quad . \quad . \quad (27)$$

The total weight W of the beam is given by

$$W = 2kM_1^n(l-a) + 2kM_2^n a \quad \ldots\ldots \quad (28)$$

which, by virtue of equations (26) and (27) becomes

$$W = 2kM_t^n l[r^n\sqrt{2r} + (1-r)^n(1-\sqrt{2r})] \quad \ldots\ldots \quad (29)$$

At minimum W,

$$\left(\frac{r}{1-r}\right)^{1-n} = \frac{(2n+1)r}{(1-2nr-r+n\sqrt{2r})} \quad \ldots\ldots \quad (30)$$

TABLE III

n	$r = M_1/M_t$	a/l
1·0	0·2946	0·2324
0·9	0·2866	0·2429
0·8	0·2777	0·2548
0·7	0·2680	0·2679
0·6	0·2571	0·2829
0·5	0·2449	0·3001

Plastic moment ratio and proportion of beam to be reinforced for beam reinforced at ends only (see fig. 6)

The most economical values of r and a/l are given in Table III for values of n from 0·5 to 1·0. The value of $r(=M_1/(M_1+M_2))$ varies from 0·2946 to 0·2449, and hence the reinforced section has a full plastic moment from 140% to 208% greater than the unreinforced section. The value of a/l varies from 0·2324 to 0·3001. A satisfactory working rule would therefore be to reinforce an eighth of the length of the beam at either end.

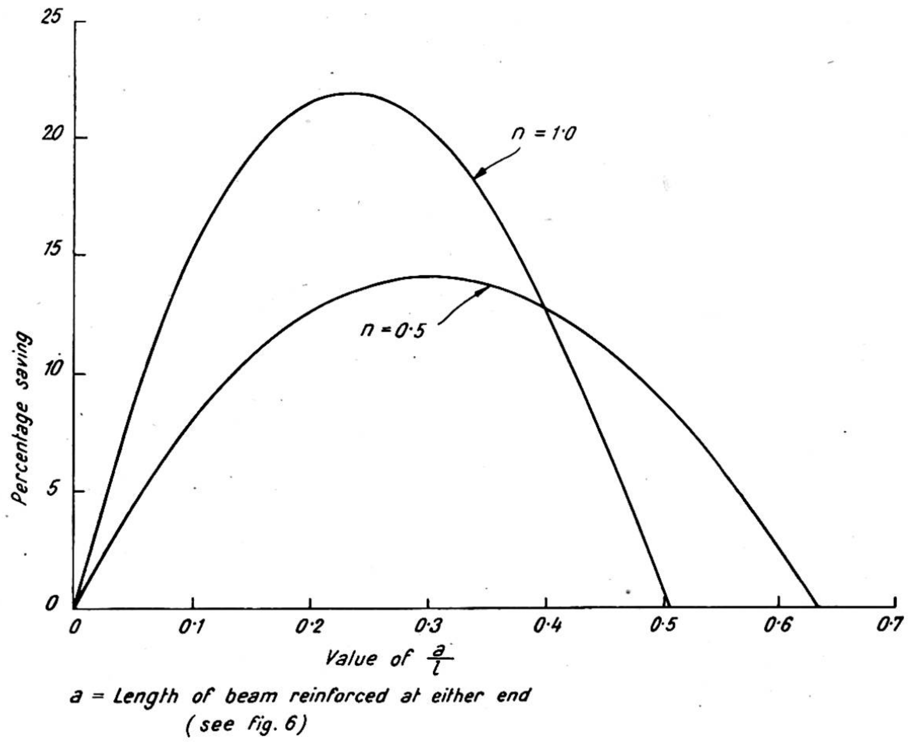

Fig. 7. Economies achieved by reinforcing the ends of a fixed-ended beam (uniformly distributed load)

The variation of the percentage saving (compared with a uniform beam) as a/l is altered is shown in fig. 7 for $n=0{\cdot}5$ and $n=1{\cdot}0$. It will be seen that by taking $a/l=0{\cdot}25$ there is very little loss in economy in either case. When $n=0{\cdot}5$ the maximum saving possible is 14·1 % and for $n=1{\cdot}0$ it is 22·0 %. These figures compare with the ideally attainable economies of 35·1 % and 50·0 % respectively.

The economy practically attainable by reinforcing the ends alone is therefore quite appreciable. It should be noted, however, that the surrounding members must provide a total moment of resistance equal to the full plastic moment of the reinforced part of the beam, and this may sometimes be a serious disadvantage.

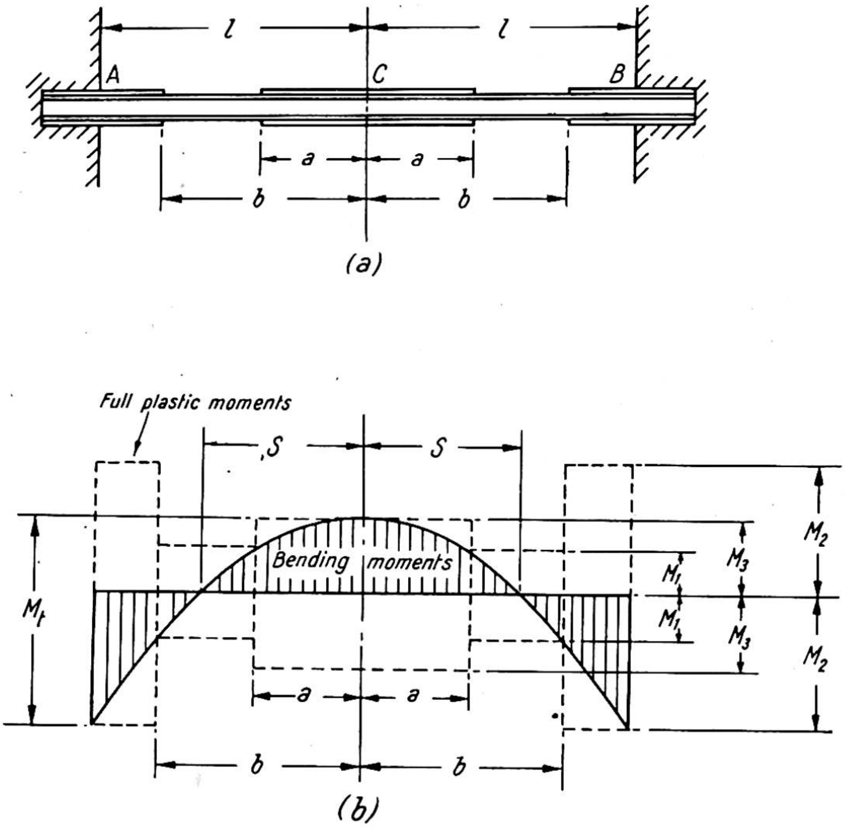

Fig. 8. Bending moment distribution for a beam reinforced at both centre and ends (uniformly distributed load)

(c) *Section increased at both centre and ends*

The advantage obtained by reinforcing both centre and ends may be estimated by considering the beam shown in fig. 8(a). This is reinforced for a distance a either side of the centre and at each end for a distance $(l-b)$. The bending moment diagram, shown shaded in fig. 8(b), is completely enclosed by the graph of the full plastic moments (shown dotted). The unreinforced section has a moment of resistance M_1, the ends a moment of resistance M_2 and the centre a moment of resistance M_3.

Hence
$$M_3-M_1=\frac{a^2}{l^2}M_t \quad \ldots \ldots \ldots \quad (31)$$

$$M_3+M_1=\frac{b^2}{l^2}M_t \quad \ldots \ldots \ldots \quad (32)$$

$$M_2+M_3=M_t \quad \ldots \ldots \ldots \quad (33)$$

Solving for M_1, M_2 and M_3,

$$M_1=\frac{b^2-a^2}{2l^2}M_t \quad . \quad . \quad . \quad . \quad . \quad (34)$$

$$M_2=\frac{2l^2-a^2-b^2}{2l^2}M_t \quad . \quad . \quad . \quad . \quad . \quad (35)$$

$$M_3=\frac{a^2+b^2}{2l^2}M_t \quad . \quad . \quad . \quad . \quad . \quad (36)$$

The total mass of the beam thus becomes

$$W=2^{1-n}k\frac{M_t^n}{l^{2n}}[a(a^2+b^2)^n+(b-a)(b^2-a^2)^n+(l-b)(2l^2-a^2-b^2)^n] \quad . \quad (37)$$

When W has its minimum value, $\partial W/\partial a=0$ and $\partial W/\partial b=0$.

When $n=1{\cdot}0$ the above conditions give $3a^2-al=0$ and $3b^2-bl-l^2=0$,

whence

$$\frac{a}{l}=\tfrac{1}{3}=0{\cdot}3333$$

$$\frac{b}{l}=\frac{1+\sqrt{13}}{6}=0{\cdot}7676$$

The moment of resistance is 171·8 % greater than that of the unreinforced beam at the ends, and 46·5 % greater at the centre. The saving is 25·7 %, compared with 22·0 % with end reinforcement only and 3·7 % with central reinforcement only.

When $n=0{\cdot}5$, it is found that

$$\frac{ab}{\sqrt{a^2+b^2}}+\frac{(b-a)(a+2b)}{\sqrt{b^2-a^2}}-\frac{2l^2+lb-a^2-2b^2}{\sqrt{2l^2-a^2-b^2}}=0$$

$$\frac{2a^2+b^2}{\sqrt{a^2+b^2}}-\frac{(b-a)(2a+b)}{\sqrt{b^2-a^2}}-\frac{a(l-b)}{\sqrt{2l^2-a^2-b^2}}=0$$

These equations give $a/l=0{\cdot}3185$ and $b/l=0{\cdot}7074$. The moments of resistance at the ends and centre are respectively 250·5 % and 50·9 % greater than that of the unreinforced beam. The saving is 16·1 %, compared with 14·1 % with end reinforcement only and 2·0 % with central reinforcement only.

Hence the percentage saving with both central and end reinforcement is very little greater than the saving with end reinforcement only.

5. Conclusions

It has been shown that the adoption of beams of varying cross-section can lead to considerable economies in total material consumption when the basis of design is the ultimate load which the beam will carry as calculated by the simple plastic theory. The best shape for the beams has been calculated for the case of a fixed-ended beam carrying a uniformly distributed load, the minimum cross-sections occurring at about one-fifth the length of the beam from the centre. The maximum theoretical economies are of the order 35–50 %.

Since the construction of a beam of continuously varying cross-section may have considerable practical disadvantages, an investigation has been made into the effect of reinforcing either the centre or the ends of the beam, or both centre and ends simultaneously. It has been shown that there is only a negligible advantage in reinforcing the centre, but that reinforcing the ends does lead to appreciable economies. The economy achieved by reinforcing both centre and ends is virtually no greater than that achieved by reinforcing the ends alone.

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Summary

The simple plastic theory gives a direct means of determining the form of a fixed-ended beam of varying cross-section such that the total weight of material shall be an absolute minimum. The paper shows how this form may be deduced for a uniformly distributed load, both when the cross-section of the beam can be varied continuously, and when the size of the beam can only be adjusted in discrete intervals. The maximum theoretically attainable economies of material are discussed.

Résumé

La théorie simple de la plasticité fournit un moyen direct pour déterminer la forme à donner à une poutre encastrée à ses extrémités et présentant une section non uniforme, pour que le poids total de métal employé constitue un minimum absolu. L'auteur montre comment l'on peut déterminer une telle forme dans le cas d'une charge uniformément répartie, aussi bien lorsque la section de la poutre peut varier d'une manière continue que lorsque ses dimensions effectives ne peuvent être choisies que dans des intervalles déterminés. Il discute l'économie maximum de métal que l'on peut réaliser du point de vue théorique.

Zusammenfassung

Die einfache Plastizitätstheorie erlaubt uns die direkte Bestimmung derjenigen Form eines eingespannten Balkens mit veränderlichem Querschnitt, bei der das Gesamtgewicht des Materials ein absolutes Minimum sein soll. Der Aufsatz zeigt die Ermittlung dieser Form bei gleichmässig verteilter Belastung, einerseits, wenn der Querschnitt des Balkens stetig veränderlich ausgeführt werden kann und andererseits, wenn seine Abmessungen nur in bestimmten Abstufungen verändert werden können. Die höchste theoretisch mögliche Ausnützung des Materials wird untersucht.